Instructions: Please solve the following problems. Work on your own and do not discuss these problems with your classmates or anyone else.

1. Is the following statement true or false (justify your answer)?
   "Let $A$ be a subset of $\mathbb{R}^n$ and $\text{Int}(A)$ the set of all interior points. Then $\text{Int}(A) = \emptyset$ if and only if $\mu^*(A) = 0$. (Here $\mu^*$ denotes the outer measure)."

2. Let $f : [0, 1] \to \mathbb{R}$ be a twice differentiable function.
   (i) Is $f''$ a continuous function? Justify your answer.
   (ii) Is $f''$ a Lebesgue measurable function? Justify your answer.

3. Let $f \in L^1([0, 1])$. Find
   \[
   \lim_{k \to \infty} \int_{[0,1]} k \ln \left(1 + \frac{|f(x)|^2}{k^2}\right) d\mu,
   \]
   where $\mu$ stands for the Lebesgue measure in $\mathbb{R}$. Justify your answer.

4. Let $f : [a, b] \to \mathbb{R}$ be a $C^1$-function (that is $f$ is differentiable and $f'$ is continuous). Show that
   \[
   \text{Var}_a^b[f] = \int_a^b |f'(x)| dx
   \]
   (here $\text{Var}_a^b[f]$ stands for the variation of $f$ on $[a, b]$).
Problem 1. Let $\mathbb{E}$ be a Banach space and $A, B \subset \mathbb{E}$ two sets. Under the assumption that $A$ is totally bounded show that

$$\overline{A + B} = \overline{A} + \overline{B}.$$
Problem 2. Let $\mathbb{E} := L^1([a, b], \mu)$, where $\mu$ stands for the Lebesgue measure on the interval $[a, b]$. Define $A : \mathbb{E} \to \mathbb{E}$ by

$$(Af)(x) := \int_a^x f(t) \, dt, \quad f \in \mathbb{E}.$$ 

(a) Show that $A$ is a bounded linear operator.

(b) Compute the norm $\|A\|$.
Problem 3. Let $\varphi : E \to (-\infty, \infty]$ be a function such that $D(\varphi) \neq \emptyset$.

(a) Give a definition of the conjugate function $\varphi^*$.

(b) For a fixed $x_o \in E$ and $\varphi : E \to \mathbb{R}$ given by $\varphi(x) = \|x - x_o\|, x \in E$, compute $\varphi^*$. 
Problem 4. Let $\mathbb{E}$ and $\mathbb{F}$ be two Banach spaces and $T \in L(\mathbb{E}, \mathbb{F})$. Show that if $\mathbb{E}$ is reflexive then the set

$$\{T(x) : x \in \mathbb{E}, \|x\| \leq 1\}$$

is closed in $\mathbb{F}$. 
Complex Analysis Qualifying Exam

Spring 2020
January 10, 2020

1. **[25 points]** True or false (Justification is needed):
   
   (a) There does not exist an analytic function \( f(z) = u(x, y) + iv(x, y) \) for which \( u(x, y) = y^3 + 5x \).

   (b) Suppose a function \( f \) has a Taylor series representation centered at \( z_0 \). Then there exists an \( R>0 \) so that \( f \) is analytic everywhere inside the circle of convergence \( |z-z_0|=R \), and is not analytic everywhere outside \( |z-z_0|=R \).

   (c) There exists a fractional-linear transformation which maps the points \( 0, -1, i, \) and \( 1 \) onto the points \( -i, \infty, 0, \) and \( (1-i)/2 \) respectively.

2. **[25 points]** Define \( f(z) = e^z (\cos y + i \sin y) \).

   (a) Show that \( f(z) \) is the unique function analytic on \( C \) that satisfies \( f'(z) = f(z), f(0) = 1 \).

   (b) Conclude from (a) that \( f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \).

3. **[25 points]**
   
   (a) State the Schwarz Lemma.

   (b) Let \( f : D \to D \) be a holomorphic map from the unit disk \( D \) to itself. Prove that for all \( z \in D \):

   \[
   \frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}.
   \]

4. **[25 points]** Use the calculus of residues to evaluate the integral

\[
\int_{0}^{\infty} \frac{x^2}{x^6 + 1} \, dx.
\]

Verify all steps of the calculation.
Problem 1 (25 points).

(a) (5 points) State the fundamental theorem of finitely-generated abelian groups.

(b) (20 points) Determine the structure (as a direct product $\mathbb{Z}^r \times \bigotimes_{i=1}^k \mathbb{Z}/n_i\mathbb{Z}$ with $n_i|n_{i+1}$) of the finitely-generated abelian group generated by three elements $x, y, z$ subject to the relations

\begin{align*}
2x + 3y + z &= 0 \\
x + 2y + 2z &= 0 \\
2x + 2y + 3z &= 0.
\end{align*}
Problem 2 (25 points).
(a) (5 points) State the Sylow theorems.

(b) (20 points) Let $G$ be a group of order 72. Show that $G$ is not simple.
Problem 3 (25 points). For $x, y \in G$, write $[x, y] = x^{-1}y^{-1}xy$ for the *commutator* of $x$ and $y$. Let $N = \langle [x, y] : x, y \in G \rangle$ be the group generated by the commutators of all elements of $G$. Prove that $N \trianglelefteq G$ and that $G/N$ is abelian.
Problem 4 (25 points). Let $G$ be a finite group acting on a finite set $X$.

(a) (5 points) State Burnside’s lemma for the number of orbits $|X/G|$.

(b) (20 points) Suppose now that $G$ is nontrivial and that $|X| > 1$. If $|X/G| = 1$, prove that some $g \in G$ has no fixed points.
Bonus (10 points). Prove that the group with presentation \( \langle a, b : ab = b^a, ba = a^2b \rangle \) is trivial.
Problem 1) (15 pts) Find the principal matrix solution at $t_0 = 0$ for the following system.

$$
\begin{align*}
x'_1 &= 3x_1 - x_2 \\
x'_2 &= x_1 + x_2
\end{align*}
$$
Problem 2) (15 pts) Find the solution to the following differential equation.

\[tx'' + 2x' - \frac{2}{t}x = 9 \quad x(1) = 1 \quad x'(1) = 2\]

(Hint: Try some functions to find the first homogeneous solution)
Problem 3) (20 pts) Prove or give a counterexample for the following statements. Verify your answer.

\[ x' = f(x) \quad x(0) = x_0 \]

3a) If \( f \) is continuous (\( C^0 \)), then there exists a unique solution.

3b) If \( f \) is differentiable (\( C^1 \)), then there exists a unique solution.
Problem 4) (15 pts) Consider the following ODE: \( x' = t - \sqrt[3]{x} \)

4a) Discuss the limit of the solution \( x(t) \) when \( t \to \infty \) for a given initial condition \( x(t_0) = x_0 \).

4b) Is there any solution where \( |x(t)| \to \infty \) in finite time?
Problem 5a) (9 pts) Find the normalized eigenfunctions of the following problem.

\[ y'' + \lambda y = 0 \quad y(0) = 0 \quad y'(1) = 0 \]

5b) (6 pts) By using part a, find a function \( f(x) \) where the following equation has no solution.

\[ y'' + \frac{3\pi^2}{4} y = f(x) \quad y(0) = 0 \quad y'(1) = 0 \]
Problem 6) (20 pts) Is there any function $x(t) \in C^0([0,1], \mathbb{R})$ such that $x(t) = x^4(t) + \frac{t}{10}$

(Hint: Use contraction mapping principle.)
QE ID:

There are 5 problems. Each problem is worth 20 points. The total score is 100 points.

Show all your work to get full credits.
Problem 1. Construct 3 different atlases for $S^2 = \{ x \in \mathbb{R}^3 : ||x|| = 1 \}$. 
Problem 2. Show that $\mathbb{R}P^2$ is a smooth manifold.
Problem 3. Show that the map $f : S^2 \to \mathbb{RP}^2$ defined by $f(x_1, x_2, x_3) = [(x_1, x_2, x_3)]$ is smooth and has constant rank 2.
Problem 4. Let $\mathbb{T}^2 = S^1 \times S^1 \subset \mathbb{R}^4$ denote the 2-torus, defined as the set of points $(w, x, y, z)$ such that $w^2 + x^2 = y^2 + z^2 = 1$, with the product orientation determined by the standard orientation on $S^1$. Compute $\int_{\mathbb{T}^2} \omega$, where $\omega$ is the following 2-form on $\mathbb{R}^4$:

$$\omega = xz \, dw \wedge dy.$$
Problem 5. Let \( e^1, e^2, e^3 \) be the standard dual basis for \((\mathbb{R}^3)^*\). Show that the tensor \( e^3 \otimes e^2 \otimes e^1 \) is not equal to a sum of an alternating tensor and a symmetric tensor.
Numerical PDE Qualifying Exam January 2020

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

(1a) Given the one-way wave equation \( u_t + 4u_x = 0 \) defined on the interval \( t \geq 0 \) and \( 0 \leq x \leq 3 \) with initial data \( u(0,x) = 1 \) and boundary data \( u(t,0) = -1 \), determine the solution at the following points. (Hint: to receive credit you must draw graphs.)

(i) \( u(2,2) \)
(ii) \( u(\frac{1}{8},1) \)
(iii) \( u(\frac{1}{4},1) \)

(b) Consider the 1D heat equation \( u_t =bu_{xx}, \) with \( b > 0, \ b \in \mathbb{R}. \) Here \( u(t,x) \) is the temperature at time \( t, \) position \( x. \) Prove an order of accuracy estimate for the forward-time, central-space scheme for this equation. The scheme is:

\[
\frac{v_{m}^{n+1} - v_{m}^{n}}{k} = b \left[ \frac{v_{m+1}^{n} - 2v_{m}^{n} + v_{m-1}^{n}}{h^2} \right].
\]

(2a) Consider the one-way wave equation \( u_t + au_x = 0. \) Prove that the backward-time, backward-space scheme is consistent with this equation. The scheme is:

\[
\frac{v_{m}^{n+1} - v_{m}^{n}}{k} + a \frac{v_{m}^{n+1} - v_{m-1}^{n+1}}{h} = 0.
\]

(b) Prove stability for the backward-time, backward-space scheme.

(c) Is this scheme convergent for the one-way wave equation? Why or why not?

(3) Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \) with a smooth boundary \( \Gamma. \) Let \( H^{1}_{0} = \{ v \in H^{1}(\Omega) : \gamma(v) = 0 \}, \) where \( \gamma : H^{1}(\Omega) \to L^{2}(\Gamma) \) is the trace (bounded, linear) operator. Consider the BVP:
\[
\begin{aligned}
-\nabla \cdot (a(x) \nabla u) + b(x) \cdot \nabla u + c(x) u &= f \text{ in } \Omega, \\
u(x) &= 0 \text{ on } \Gamma,
\end{aligned}
\]
where \( f \in L_2(\Omega) \) and \( a(x), b(x) \) and \( c(x) \) are smooth functions on \( \bar{\Omega} \).

a. Find a variational formulation of this problem in \( H_0^1 \).

b. In the special case where \( b(x) \) and \( c(x) = 0 \), show existence and uniqueness of a weak solution. What conditions on \( a(x) \) have to be assumed?

(4a) Consider the 1D heat equation:

\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x, t)
\]

defined for \( 0 < x < 1 \) and \( t > t_0 \) with zero Dirichlet boundary conditions and an appropriate initial condition. Discretize the spatial derivatives in the problem using the Galerkin finite element method. Write down the system of ode’s that results from this partial discretization of the problem.

(b) In iteratively solving a matrix problem \( Au = f \) resulting from discretizing a pde, you discover that your error decreases for the first 10 iterations, but after iteration 10, the error no longer decreases very much. You plot your error which is given in the Figure below. This figure uses 12 node points. Explain what the multigrid method does to reduce the error further and why. Draw a picture to illustrate your idea.
low frequency error

\[ \sin(x) \]
MATH 6310 (TOPOLOGY) – JANUARY 2020 QUALIFYING EXAM

QE ID:

There are 5 problems. Each problem is worth 20 points. The total score is 100 points. Show all your work to get full credits.
Problem 1. Let $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ and $p = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Show that $S^n \setminus \{p\} \cong \mathbb{R}^n$ by using the stereographic projection.
Problem 2. A continuous map \( r : X \to A \), where \( A \) is a subspace of a topological space \( X \), is called a retraction if \( r(a) = a \) for all \( a \in A \). Show that there are no retractions of \( X = S^1 \times D^2 \) onto its boundary torus \( A = S^1 \times S^1 \).
**Problem 3.** Let $a$ and $b$ be the generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two $S^1$ summands. Draw a picture of the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by $a^2, b^2$ and $(ab)^2$. 
Problem 4. (a) Find a Δ-complex structure of the projective plane $\mathbb{R}P^2$. 
(b) Compute the (simplicial) homology groups of $\mathbb{R}P^2$. 
Problem 5. If a finite CW complex $X$ is the union of subcomplexes $A$ and $B$, show that $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$. Here $\chi(Y)$ denotes the Euler characteristic of $Y$. 
MATH 6319 - Ph. D Qualifying Examinations
Spring 2020
Felipe Pereira & V. Ramakrishna

**CHOICE:** Do any 4 of the Qs below. Each Q is worth 25 points.

- **Q1** Compute in closed form the eigenvalues of the matrix $A = (i + j)$. Explain your work.

- **Q2** State the Crabtree-Haynsworth formula. Derive it by using a result on the Schur complement in a threefold product of matrices.

- **Q3**
  State and prove Fischer’s inequality for positive semidefinite matrices.

- **Q4** Let $V$ be a complex pre-inner product space. Find a suitable quotient space $V/W$ and suitable inner-product on it which is related to that on $V$. Justify your answer fully (i.e., identify $W$, explain why it is a subspace; explain why your inner product is well defined and why it is indeed an inner product etc.).

- **Q5** Show that the Fourier transform of a Gaussian is another Gaussian, by setting up a differential equation for the Fourier transform.

- **Q6** Derive the CBS inequality for vectors in $\mathbb{R}^n$ by calculating the trace of a suitably concocted rank 2 matrix from these vectors.
Problem 1 (25 points). Give a combinatorial proof of the identity
\[
\binom{3n}{3} = 3 \binom{n}{3} + 6n \binom{n}{2} + n^3.
\]
Problem 2 (25 points). Prove combinatorially that
\[
\binom{n}{k} = \binom{n + k - 1}{k},
\]
where \(\binom{n}{k}\) counts the number of ways to choose \(k\) elements from an \(n\) element set with replacement.
Problem 3 (25 points).

(a) State the binomial theorem for the expansion of \((1 + x)^\alpha\), where \(\alpha \in \mathbb{C}\).

(b) Use part (a) to show that

\[ n(n + 1)2^{n-2} = \sum_{i=1}^{n} i^2 \binom{n}{i} \]

and find

\[ \sum_{j=0}^{n} \frac{1}{j+1} \binom{n}{j}. \]
Problem 4 (25 points).
(a) Define the notion of Eulerian cycle of a graph $G$.

(b) Finish the theorem: *A graph $G$ has an Eulerian cycle iff...*

(c) Find an Eulerian cycle in the graph below.
Bonus (10 points). Let $G$ be a graph with $v_G = n > 2$ and $e_G > \frac{n^2}{4}$. Show that $G$ is not a bipartite.