1. Let $A \subset \mathbb{R}^2$ be the graph of the function $f : \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} 
\sin \left( \frac{2021}{x} \right), & \text{if } x \in [-1, 1] \setminus \{0\}; \\
2020, & \text{if } x = 0.
\end{cases}$$

Is the set $A \cup \{(0, y) \in \mathbb{R}^2 : -1 \leq y \leq 1 \}$ measurable with respect to the Lebesgue measure on the plane? Justify your answer.

2. Let $f : [0, 1] \to \mathbb{R}$ be a Lebesgue integrable function and let $K \subset [0, 1]$ be a measurable subset such that $\mu(K) > 0$ and $f(x) > 0$ for each $x \in K$ (here $\mu$ stands for the Lebesgue measure). Show that

$$\int_K f d\mu > 0.$$

3. Let $f : [0, 1] \to \mathbb{R}$ be a Lebesgue integrable function. Assume that

$$\int_0^x f d\mu = 0 \quad \text{for each } x \in [0, 1].$$

Show that $f(x) = 0$ for almost all $x \in [0, 1]$.

4. Let $f : [a, b] \to \mathbb{R}$ be a $C^1$-function (that is $f$ is differentiable and $f'$ is continuous). Show that

$$\text{Var}_a^b[f] = \int_a^b |f'(x)| dx.$$

(here $\text{Var}_a^b[f]$ stands for the variation of $f$ on $[a, b]$).

Good luck!
Problem 1. Let $E$ be a Banach space and $C \subset E$ a non-empty convex set.

(a) Show that if $x \in C$ and $y \in \text{int}(C)$ then for every $t \in (0,1)$ we have

$$tx + (1-t)y \in \text{int}(C);$$

(b) Show that if $C$ is closed then $C$ is also weakly closed (with respect to the weak topology $\sigma(E,E^*)$).
Problem 2. Consider the Euclidean space $\mathbb{E} := \mathbb{R}^n$, i.e. $\mathbb{E}$ is equipped with the norm

$$
\| x \|_2 := \left[ \sum_{k=1}^{n} x_k^2 \right]^{\frac{1}{2}}, \quad x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.
$$

Suppose $C$ is an open bounded convex set such that $0 \in C$ and $C = -C$.

(a) Show that the function

$$
\| x \| = \inf \{ r > 0 : x \in rC \}, \quad x \in \mathbb{R}^n
$$

is a norm on $\mathbb{R}^n$.

(b) Show directly that there exist constants $C, c > 0$ such that

$$
\forall x \in \mathbb{E} \quad c\| x \|_2 \leq \| x \| \leq C\| x \|_2.
$$
Problem 3. Let $E$ be a reflexive Banach space and consider a continuous convex function $\varphi : E \to \mathbb{R}$ such that $\lim_{\|x\| \to \infty} \varphi(x) = \infty$. Show that

$$\exists x_0 \in E \quad \varphi(x_0) = \inf_{x \in E} \varphi(x).$$
Problem 4. Consider the Banach space $E := C([0, 1], \mathbb{R})$ (equipped with the norm $\|\varphi\|_\infty := \sup_{t \in [0, 1]} |\varphi(t)|$). Define the following linear operator $A : E \to E$ by

$$(A\varphi)(t) := \int_0^t \varphi(s)s^2 ds, \quad \varphi \in E, \ t \in [0, 1].$$

(a) Show that the linear operator $A$ is a bounded,

(b) Compute $\|A\|$. 

Complex Analysis Qualifying Exam

Summer 2020
Friday, August 7, 2020

1. [25 points] True or false (Justification is needed):
   (a) If \( f(z) \) is analytic on a domain \( D \subseteq \mathbb{C} \), and \( \gamma \) is a closed curve in \( D \), then \( \int_{\gamma} f(z) \, dz = 0 \). Is this true for any \( f(z) \) and any \( D \)?
   (b) If \( f(z) \) is analytic on the unit disk \( D = \{ z : |z| < 1 \} \), then there exists an \( a \in D \), \( a \neq 0 \) such that \( |f'(a)| \geq |f(0)| \).
   (c) Every analytic function \( f(z) \) on a domain \( D \) has a power series expansion \( f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \) for each \( z_0 \in D \) with a strictly positive radius of convergence.

2. [25 points]
   (a) Given four distinct points \( z_1, z_2, z_3, z_4 \) in \( \mathbb{C} \), their cross ratio, which is denoted \( (z_1, z_2, z_3, z_4) \), is defined to be the image of \( z_4 \) under the fractional linear transformation that sends \( z_1, z_2, z_3 \) to \( \infty, 0, 1 \), respectively. Prove that if \( \phi \) is a fractional linear transformation then \( (\phi(z_1), \phi(z_2), \phi(z_3), \phi(z_4)) = (z_1, z_2, z_3, z_4) \).
   (b) Prove that the four distinct points \( z_1, z_2, z_3, z_4 \) of \( \mathbb{C} \) lie on a (generalized) circle if and only if the cross ratio \( (z_1, z_2, z_3, z_4) \) is real.
   (c) Compute the cross ratio \( (2, -2, 2i, z) \) and use (b) to decide whether the points \( \hat{z}_{1,2} = 1 \pm i \sqrt{3} \) and \( \hat{z}_{3,4} = 2 \pm i \) lie on the circle \( |z| = 2 \).

3. [25 points]
   (a) Let \( H = \{ z \in \mathbb{C} \mid \text{Im} \, z > 0 \} \) be the upper half-plane. Let \( f : H \to \mathbb{C} \) be holomorphic and satisfy \( |f(z)| \leq 1 \) for \( z \in H \) and \( f(i) = 0 \). Show that \( |f(z)| \leq \left| \frac{z-i}{z+i} \right| \) for \( z \in H \).
   (b) Let \( \Omega \) be a bounded region, \( a \in \Omega \) and \( f : \Omega \to \Omega \) be a holomorphic map such that \( f(a) = a \). Show that \( |f'(a)| \leq 1 \).

4. [25 points] Use the calculus of residues to evaluate the integral
   \[
   \int_{-\infty}^{\infty} \frac{x^2 + 1}{(x^2 + 4)(x^2 + 9)} \, dx .
   \]
   Verify all steps of the calculation.
Problem 1 (25 points).

(a) (5 points) State the fundamental theorem of finitely-generated abelian groups.

(b) (20 points) Determine the structure (in the form of your statement of part (a)) of the finitely-generated abelian group generated by three elements $x, y, z$ subject to the relations

\begin{align*}
2x + 4y + 2z &= 0 \\
12x + 6y + 4z &= 0 \\
8x + 3y + 3z &= 0.
\end{align*}
Problem 2 (25 points). Let $\mathbb{Z}(G)$ be the group ring of $G$ over the integers (that is, the ring of formal linear combinations of elements of $G$, with coefficients in $\mathbb{Z}$). Let $I$ be a right ideal of $\mathbb{Z}(G)$ and define

$$G_I = \{ g \in G \mid (1 - g) \in I \}.$$ 

Show that $G_I$ is a subgroup of $G$, and that it is normal in $G$ if $I$ is a 2-sided ideal.
Problem 3 (25 points). Show that if $H$ is a normal subgroup of $G$, then the center $Z(H)$ of $H$ is a normal subgroup of $G$. Give an example of $H$ and $G$ such that $H$ is a normal subgroup of $G$ but $Z(H)$ is not a subgroup of $Z(G)$. 
Problem 4 (25 points).

(a) (5 points) Let $G$ be a finite group acting on a finite set $X$. State Burnside’s lemma for the number of orbits $|X/G|$.

(b) (20 points) Let $G$ be a finite group and let $g \in G$ and $h \in G$ be two randomly chosen elements (with replacement). Show that the probability that $g$ and $h$ commute is $k/|G|$, where $k$ is the number of conjugacy classes in $G$. (Hint: use part (a).)
Bonus (10 points). Let $p$ and $q$ be primes with $p > q$ and $q \nmid (p - 1)$. Show that all groups of order $pq$ are cyclic.
Problem 1) (15 pts) Find the principal matrix solution at $t_0 = 0$ for the following system.

\[
\begin{align*}
    x_1' &= 2x_1 - x_2 \\
    x_2' &= x_1 + 4x_2
\end{align*}
\]
Problem 2) Solve the following initial value problems:

2a) (10 pts) \[ 8tx' + 12x^2 = -4t \] \[ x(1) = 1 \]

2b) (10 pts) \[ x' = 3t^2 + 2\sqrt{x-t^3} \] \[ x(0) = 4 \]
**Problem 3** (15 pts) Find the solution to the following differential equation.

\[ t^2x'' + tx' - 4x = \frac{12}{t} \quad x(1) = 1 \quad x'(1) = 2 \]

*(Hint: \( x = t^2 \) is a homogeneous solution)*
Problem 4) (15 pts) Prove or give a counterexample for the following statement:

Let \( x' = f(t, x) \) with \( x(t_0) = x_0 \). If \( f \in C^1(\mathbb{R}^2, \mathbb{R}) \), then there exists a unique solution \( \phi(t) \) whose domain is \( \mathbb{R} \).

[Here, \( C^1(\mathbb{R}^2, \mathbb{R}) \) represents at least once differentiable functions whose domain is \( \mathbb{R}^2 \).]
Problem 5) (20 pts) Consider the following ODE: \( x' = t - \ln x \)

5a) Discuss the limit of the solution \( x(t) \) when \( t \to \infty \) for a given initial condition \( x(t_0) = x_0 > 0 \).

5b) Is there any solution where \( |x(t)| \to \infty \) in finite time?
Problem 6a) (9 pts) Find the normalized eigenfunctions of the following problem.

\[ y'' + \lambda y = 0 \quad y'(0) = 0 \quad y(\pi) = 0 \]

6b) (6 pts) By using part a, find a function \( f(x) \) where the following equation has no solution.

\[ y'' + \frac{9}{4}y = f(x) \quad y'(0) = 0 \quad y(\pi) = 0 \]
Qualifying Exam
Math 7329  August 2020
Topological and algebraic methods in nonlinear DEs

QE ID_________________________

Instructions: Please solve the following problems. Work on your own and do not discuss these problems with your classmates or anyone else.

1. Given a matrix
\[
A = \begin{pmatrix}
a & 0 & 0 & b \\
0 & c & d & 0 \\
0 & -d & c & 0 \\
-b & 0 & 0 & a \\
\end{pmatrix}
\]
and a vector \(v = (1, 2, 3, 4)^t\), under which conditions with respect to the parameters \(a, b, c\) and \(d\) does the system
\[
\dot{x} = Ax + \cos^{2020}(2t)v \quad (x \in \mathbb{R}^4)
\]
admit a periodic solution? Justify your answer.

2. Does the system
\[
\begin{align*}
\dot{x} &= 4x^3 + 2xy^2 + 2xz^2 + (3y^2 + 4z^2 + 2020) \cos(2t) \\
\dot{y} &= 2x^2y + 4y^3 + 2yz^2 + (4x^2 + 11z^2) \sin(2t) \\
\dot{z} &= 2x^2z + 2zy^2 + 4z^3 + (8x^2 + 4y^2) \cos(2t)
\end{align*}
\]
admit a periodic solution? Justify your answer.

3. Let \(K_1\) and \(K_2\) be two compact sets in \(\mathbb{R}^2\). Show that there exist \(a, b, c \in \mathbb{R}\) such that:
\[
\mu\{(x, y) \in K_1 : ax + by \geq c\} = \mu\{(x, y) \in K_1 : ax + by \leq c\}
\]
and
\[
\mu\{(x, y) \in K_2 : ax + by \geq c\} = \mu\{(x, y) \in K_2 : ax + by \leq c\}
\]
(here “\(\mu\)” stands for the Lebesgue measure on the plane).

4. Let \(D \subset \mathbb{C}\) be the unit disc. Show that any two continuous paths \(\gamma_1\) and \(\gamma_2\) in \(D\) such that \(\gamma_1\) connects 1 and \(-1\) and \(\gamma_2\) connects \(i\) with \(-i\), must intersect in \(\overline{D}\).

Good luck!
Problem 1 (25 points).
(a) (5 points) State the Havel-Hakami Theorem, giving a condition for a degree sequence to be graphic.

Use the Havel-Hakami Theorem to determine which of the following degree sequences are graphic. If the degree sequence is graphic, draw a graph $G$ with that degree sequence.
(b) (5 points) $d_1 = (3, 3, 3, 3, 2)$

(c) (5 points) $d_2 = (4, 3, 3, 2, 1)$

(d) (5 points) $d_3 = (4, 4, 3, 2, 1)$

(e) (5 points) $d_4 = (2, 2, 2, 1, 1)$
Problem 2 (25 points). Prove that if $G$ is a planar graph, then $v - e + f = 2$, where $v$ is the number of vertices of $G$, $e$ is the number of edges, and $f$ is the number of faces.
Problem 3 (25 points). Let $F_n$ denote the number of ways to climb $n$ steps using one or two steps at a time.

(a) (10 points) Show that $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

(b) (15 points) Show that the ordinary generating function for the sequence $(F_n)_{n=0}^\infty$ is $F(x) = \frac{1}{1-x-x^2}$.
Problem 4 (25 points). Give a combinatorial proof that $\sum_{k=0}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}$. (Non-combinatorial proofs will only receive partial credit.)
Bonus (10 points). How many binary sequences \((\epsilon_1, \epsilon_2, \ldots, \epsilon_n)\) (with \(\epsilon_i \in \{0, 1\}\)) satisfy
\[\epsilon_1 \leq \epsilon_2 \geq \epsilon_3 \leq \epsilon_4 \geq \epsilon_5 \leq \cdots?\]