Licensing Mechanisms for Product Lines

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Abstract

This paper examines the effect of different types of licensing payments on the strategies and profits of contracting parties when a licensee pays royalties on technology it uses to develop a product line. I compare three types of royalties: 1) per-unit royalties; 2) ad valorem royalties; and 3) ad valorem royalties with a cap, where the licensor uses an ad valorem rate but places a cap on price to which this rate is applied. Consumers differ in their willingness to pay for quality, and the licensee cannot distinguish among them. I find that per-unit contracts leave the quality levels the same as in the case without royalties, but a larger per-unit rate leads to a decrease in market coverage and higher prices. In contrast, ad valorem contracts lead to lower quality levels. Ad valorem contracts with a cap further deteriorate the lower quality products, but keep the higher quality products at the same quality level as in the case without royalties. For an intermediate dispersion in quality preferences between the consumer groups and sufficient number of low-type consumers, all parties prefer per-unit contracts. For a small dispersion in quality preferences, the licensee prefers an ad valorem contract while the licensor prefers a per-unit contract. These preferences also hold for a large dispersion in quality preferences if there are only two consumer types. However, if there is a continuum of consumer types, the licensor prefers an ad valorem rate with a cap while the licensee prefers a per-unit rate.

Keywords: licensing; product line pricing; standard-essential patents; per-unit royalty; ad valorem royalty
1 Introduction

Consider a licensor who owns a standard-essential patent (SEP). It has to negotiate a licensing contract with a licensee who will employ this technology to develop a line of products that vary in quality. These quality differences are along the dimensions that are unrelated to licensor’s technology. For example, Apple uses Qualcomm’s modem chips to produce a variety of phones differentiated by display size and quality, camera characteristics, battery life, storage capacity, etc. Lenovo uses Intel’s microprocessors to produce an assortment of laptops differentiated by memory, storage, graphics, display, weight, etc. Daimler uses Nokia’s cellular technology for its line of Mercedes-Benz connected cars that differ in dimensions, horse power, engine displacement, etc.

There are two main types of payment structures in SEP licensing agreements. One involves per-unit royalties, where a licensee pays to a licensor a fixed amount for each unit it sells. Another agreement involves ad valorem royalties, where a licensee’s payment to a licensor for each unit is proportional to the price of a final product. Thus, under per-unit agreements, a licensee’s total payment is proportional to the number of units sold whereas under ad valorem agreements, it is proportional to the licensee’s total revenue.

Recent legal battle between Apple and Qualcomm has highlighted the tensions that

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1 In 2018, Nokia disclosed that it set a flat per-unit licensing fee of €3 for its 5G essential patents (Horwitz 2018).

2 For example, for the use of its wireless communications and video coding technology, Motorola charged Microsoft 2.25% of the price of the end product such as Xbox, PCs, or laptops. (Findings of Fact and Conclusions of Law in Case No. C10-1823JLR available at https://www.essentialpatentblog.com/wp-content/uploads/sites/64/2013/04/2013.04.25-D.E.-681-Findings-of-Fact-and-Conclusions-of-Law-setting-RAND-royalty1.pdf).
could exist between the cosignatories of licensing agreements with regard to the payment structure. Qualcomm’s usage of ad valorem royalties for its modem chips led to Apple’s objections.\(^3\) In 2019, the United States District Court for the Northern District of California in a suit brought by the Federal Trade Commission decided that Qualcomm’s modem chips do not drive the handset value, and, thus, Qualcomm "is not entitled to a royalty on the entire handset."\(^4\) In 2020, the Ninth Circuit Court of Appeals overturned this ruling, noting that "...law and current practice run counter to the district court’s conclusion that patent royalties cannot be based on total handset price...".\(^5\)

As firms negotiate new licensing agreements and antitrust authorities continue to scrutinize them, it would be valuable to develop a theoretical foundation illustrating the effects of different types of licensing contracts in a scenario where a licensee develops a line of products. Does, as Qualcomm litigation seems to imply, a licensor always benefit from setting ad valorem royalties whereas a licensee prefers per-unit royalties? How do different arrangements alter the quality levels and the length of the product line of the licensee? What is the impact on the final prices of the products? Which licensing mechanism is more favorable to consumers? The current paper addresses these questions.

\(^3\)For example, during Apple’s Q1 2017 earnings call, its CEO Tim Cook offered the following criticism of the royalty agreement Apple had with Qualcomm: "They [Qualcomm] were insisting on charging royalties for technologies that they had nothing to do with, and so we were in a situation where the more we innovated with unique features—like Touch ID, or advanced displays, or cameras just to name a few—the more money Qualcomm would collect for no reason... It’s somewhat like buying a sofa and you charge somebody a different price depending on the price of the house that it goes into." (Retrieved on Sep 27, 2019 from seekingalpha.com/article/4041266-apple-aapl-q1-2017-results-earnings-call-transcript)


In my model, a licensor owns a standard-essential patent for technology necessary to manufacture a product. The licensor does not produce the product, but can license this technology to a licensee, choosing one of two possible mechanisms. Under per-unit licensing, the licensee pays a fixed fee for each unit it sells to consumers. Under ad valorem licensing, the licensee’s payment for each unit sold is proportional to the price of this unit. The licensee uses the technology to manufacture products with different quality levels. In addition to payments to the licensor, the licensee incurs production costs that are larger for a higher quality product. In the base model, there are two groups of consumers that differ in their willingness to pay for quality. The licensee chooses the quality levels and prices for its two products: a high quality product targeting the high-type consumers and a low quality product targeting the low-type consumers.

I find that under a per-unit agreement, the qualities are set at the same levels as for the baseline case without any licensing payments. Remarkably, if we keep the quality preference of the low-type consumers fixed and if both groups are served, the licensor prefers the quality preference of the high-type consumers to be lower. This happens because the licensor tailors its per-unit rate to the profit from the low-type consumers. When the quality preference of the high-type consumers is relatively large, the licensee has an opportunity to extract a large surplus from this segment with a high price. In order to do so, it has to lower the quality level for the low-type consumers to prevent the high-type consumers from switching to the low quality option. The licensee then charges this segment a lower price, which leads to a lower per-unit royalty rate and a lower profit for the licensor. Since an increase in the
quality preference of the high-type consumers leads to an increase in channel profit and a decrease in the profit of the licensor, the licensee’s profit increases significantly.

Under an ad valorem agreement, a portion of the sale price is paid to the licensor. This decreases the licensee’s incentives to invest in quality and leads to deterioration of the quality levels of both products. The licensee also lowers both prices. Since the licensor’s profit is proportional to the sale prices, it has an incentive to keep those prices high by setting a relatively low ad valorem royalty rate. Given smaller channel profits under an ad valorem rate and the licensor’s inefficiency in extracting them, the licensor always prefers per-unit contracts. The preferences of the licensee depend on the model parameters. If the quality preferences of the two consumer segments are similar, a contract with a per-unit rate targeting the profit from the low-type consumers also extracts a large portion of the profit from the high-type consumers. The licensee then prefers an ad valorem contract. When the quality preferences of the two groups are sufficiently dissimilar, a per-unit rate targeting the low-type consumers leaves a lot of surplus from the high-type consumers. This leads to the licensee preferring a per-unit contract. However, if the difference in quality preferences between the two groups becomes too large, the low-type consumers are not served. Then, a per-unit rate extracts all surplus from the high-type consumers, and the licensee again prefers an ad valorem contract.

The consumer welfare in this setting coincides with the welfare of the high-type segment—the low-type consumers are either not served or their surplus is fully extracted by the firms. The surplus of the high-type consumers is proportional to the quality level of the low qual-
ity product. Thus, ad valorem contracts that distort that quality downwards decrease the welfare of the high-type consumers, and they prefer per-unit contracts as long as both groups of consumers are served. This happens for low and intermediate differences in quality preferences between the two groups. In particular, in the region where both firms prefer per-unit contracts, the high-type consumers also do so. However, for larger differences in quality preferences, a per-unit contract leads to exclusion of the low-type consumers and full surplus extraction from the high-type consumers whereas an ad valorem contract still allows for both segments to be served. In that region, the high-type consumers prefer an ad valorem contract.

The last licensing agreement I study is a modified ad valorem mechanism, where the licensor sets an ad valorem rate but places a cap on price to which this rate is applied. The optimal choice of such cap will lead to the licensee setting the efficient quality level for the high-type consumers. A high price that the licensee charges to this segment allows for a large channel profit. The licensor sets its cap so that it captures most of this profit, leaving just enough to the licensee to prevent it from setting the price of the high quality product below the cap. When the difference in quality preferences between the two groups is moderate, this strategy becomes optimal for the licensor, but is the least preferred by the licensee and the consumers.

I conclude the paper by considering a continuum of consumer types with a uniform dis-

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6 As an example, Qualcomm’s 5G licensing agreements have an ad valorem rate set at 3.25 percent of a multi-mode 5G phone price which is capped at $400 (Horwitz 2018). Thus, Qualcomm will be paid at most $13 for each phone a licensee sells.

2 Related Literature

The structure of my model follows the specification from Mussa and Rosen’s (1978) product line framework for the case of continuum consumers and from Moorthy (1984) for the two-
type case. The standard result on the optimal design of a product line is that the high-type consumers are offered the efficient quality level and pay less than their valuation for this product. Other consumers are offered suboptimally low quality levels, and the lowest type consumers have to pay a price equal to their valuation. Anderson and Dana (2009) characterized the conditions on consumer preferences and firm costs under which such price discrimination is profitable. Villas-Boas (1998) used this framework to study product line decisions in a vertical channel. He found that in comparison with the coordinated channel outcome, a manufacturer will further increase the quality differences among its products by keeping the efficient quality level for the high-end product, but decreasing the quality of the low-end product. In his model, it is the upstream firm that chooses the quality levels within the product line. Whereas this is a natural assumption in a manufacturer/retailer setting, it does not match the licensing scenario, where a licensor offers a single standard component of final goods, and it is the licensee who chooses the quality levels of several products, each using this same component. The focus of my work is on the latter scenario. In fact, the contribution of my paper to the literature on product lines in vertical channels is not limited to patent licensing, but also covers any setting where an input supplier provides a single part to a manufacturer who uses this part to make several products of varying quality levels.

Early work comparing different licensing arrangements has focused on contrasting fixed fees and per-unit royalties. A seminal paper by Kamien and Tauman (1986) examined licensing of a cost-reducing innovation to multiple producers engaged in Cournot compe-
tion. They found that a fixed fee was preferred over a per-unit royalty by the licensor and consumers. Wang (1998, 2002) showed that if a licensor also produces the final product and competes with its licensees, it would prefer a per-unit royalty over a fixed fee. A similar conclusion is reached if the producers compete in prices. Muto (1993) examined competition in prices among differentiated products and found that a licensor would gain more profit by using a per-unit royalty rather than a fixed fee if a cost-reducing innovation is relatively small.

At the same time, empirical observations of real-world licensing arrangements revealed the popularity of ad valorem royalty rates which were based on the final price of the product. Bousquet et. al. (1998) reported that 78% of the contracts of a French telecommunications firm included royalties, with 96% of those using ad valorem schemes. Sidak (2014) stated that "voluntary licenses negotiated for patented technologies implemented in multi-component products typically use the entire market value of the downstream product as the royalty base" (pg. 996). Indeed, out of 12 companies with publicly disclosed royalty rates on their LTE (Long Term Evolution, which is a wireless broadband communication standard) portfolios, nine used ad valorem rates (Armstrong et. al. 2014).

The realization that real-world contracts frequently use ad valorem rates motivated theoretical work on this topic. Most of the research that followed has focused on the scenario of a cost-reducing innovation by an internal patentee, i.e., a licensor who also produces the product. In this basic set-up, a licensor prefers an ad valorem royalty rate if the firms engage in Cournot competition (San Martin and Saracho 2010), but if the
competition is in prices, it prefers a per-unit rate combined with a fixed fee (Colombo and Filippini 2015). If a licensor does not know the size of the cost reduction for the licensee, it will either offer a separating contract using per-unit rates or an ad valorem contract that excludes a high-cost rival (Heywood et. al. 2014). If the antitrust authorities require the final product price to remain at or below the pre-licensing level, the optimal type of the contract depends on whether the licensor or the licensee is more efficient in using the innovation (Fan et. al. 2018).

However, the set-up in the above models does not cover a frequently occurring situation when a licensor does not produce a final product, but has patented an innovation that is vital for the production of this product. This scenario is examined by Llobet and Padilla (2016) who compared ad valorem and per-unit royalties when one or several complementary innovators license their essential technology to a downstream producer. They show that under many circumstances, using ad valorem royalties benefits the licensors and the consumers whereas the licensee prefers per-unit royalties. The appeal of ad valorem contracts was also illustrated by Hagia and Wright (2019) who found that in a model with unobserved (by the licensor) demand shocks and channel investments by both parties, a licensor can use price-dependent contracts to achieve the same second-best outcome as if it were able to observe the shocks and set the final price. However, with a focus on production of a single good, the extant literature on licensing contracts does not offer any insights into issues related to production of a product line. The current paper fills this gap.

The dispute over which royalty arrangement is appropriate has recently entered a legal
domain. In 2015, the Institute of Electrical and Electronics Engineers (IEEE)—a standard-setting organization—proposed to update its patent policy, which, among other things, clarified its definition of a "reasonable rate" for a patent. While acknowledging that the policy does not preclude the parties from negotiating terms mutually agreeable to both, it recommends that a reasonable rate should be based on the "smallest saleable Compliant Implementation that practices an Essential Patent Claim"\(^7\). This advocates calculating royalty rate based on the price of the component, i.e., using a per-unit royalty. The Antitrust Division of the U.S. Department of Justice issued a business review letter supporting the proposed updates, which went into effect shortly thereafter.\(^8\) These developments were criticized in the legal literature as potentially facilitating collusion among licensees (Sidak 2015) and reducing licensors’ incentives to innovate (Teece 2015).

The comparisons of effects of ad valorem versus per-unit rates also arise in other contexts. In taxation literature, Suits and Musgrave (1953) examined different types of sales taxes in a monopoly setting and concluded that ad valorem taxes resulted in a higher tax revenue than unit taxes. Additionally, if the same tax revenue is raised through two mechanisms, the price would be lower under ad valorem taxes. Delipalla and Keen (1992) extended this analysis to the case of oligopoly and found that ad valorem taxes lead to lower prices and higher tax revenue. In the context of platforms, Shy and Wang (2011) showed that payment card networks benefit from charging proportional fees while mer-

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\(^7\)IEEE SA Standards Board Bylaws, Section 6.1, available at standards.ieee.org/about/policies/bylaws/sect6-7.html

\(^8\)The letter is available at justice.gov/sites/default/files/atr/legacy/2015/02/02/311470.pdf
chants prefer fixed per-transaction fees. Wang and Wright (2017) found that ad valorem fees allow platforms to efficiently price discriminate when buyers’ value of a transaction varies proportionally with the cost, making such fees preferred over fixed per-transaction ones.

3 Basic Set-up

Consider a setting where one firm, a licensor, holds a patent on technology that is essential for production of a good. The licensor does not produce the good itself, but instead licenses its technology to another firm, a licensee. The licensee can use this technology to produce several products with different levels of quality $q$. In addition to royalty payments, the production cost of one unit of a product with quality level $q$ is $cq + d_q^2$. A consumer with taste parameter $\theta$ obtains utility $\theta q - p$ from purchasing a product of quality $q$ at price $p$. Assume that there are two types of consumers: low type with taste parameter $\theta_1$ and high type with taste parameter $\theta_2$, where $\theta_2 > \theta_1$.9 The size of the market is normalized to one, and the proportion of low-type consumers is $\lambda$.

If the licensee did not have to pay royalties, it would solve a standard product line pricing problem. The licensee has to choose two quality levels, $q_1$ and $q_2$, and charge corresponding prices $p_1$ and $p_2$ in order to maximize its profit $\lambda \left( p_1 - cq_1 - d_{q_1}^2 \right) + (1-\lambda) \left( p_2 - cq_2 - d_{q_2}^2 \right)$.

There are two individual rationality (participation) constraints: $\theta_1 q_1 - p_1 \geq 0$ and $\theta_2 q_2 - p_2 \geq 0$ as well as two incentive compatibility (self-selection) constraints: $\theta_1 q_1 - p_1 \geq \theta_1 q_2 - p_2$.

9In Section 7, I consider the case of continuum consumer types.
and $\theta_2 q_2 - p_2 \geq \theta_2 q_1 - p_1$. The solution to this problem is standard and I will briefly summarize the key points. All surplus is extracted from the low-type consumers: $p_1 = \theta_1 q_1$. The high-type consumers are indifferent between the two options: $p_2 = \theta_2 q_2 - \theta_2 q_1 + p_1 = \theta_2 q_2 - (\theta_2 - \theta_1) q_1$. We substitute these prices into the licensee’s profit function and maximize it with respect to $q_1$ and $q_2$. The efficient level of quality is offered to the high-type consumers, $q_2 = \frac{\theta_2 - c}{d}$, whereas the low-type consumers get lower than the efficient quality $q_1 = \frac{\theta_1 - c}{d} - \frac{1-\lambda}{\lambda d} (\theta_2 - \theta_1)$. If $q_1 \leq 0$, then the licensee will serve only the high-type consumers, setting the efficient quality level $q_2 = \frac{\theta_2 - c}{d}$ and extracting all their surplus with $\hat{p}_2 = \theta_2 q_2$.

Now, consider a scenario where the licensee has to pay royalties to the licensor in order to use its essential technology for the production of the product. I study a two-stage game, in which the licensor chooses its royalty rate in the first stage. The licensee then sets its product line in the second stage. I will search for the Subgame-Perfect Nash Equilibrium (SPNE) of this game and use it to study the effects of different licensing agreements on firms’ profits and on the structure of the product line. In the next section, I examine the case when the licensing contract uses per-unit royalty rates.

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10 The efficient quality level for type $\theta_i$ consumers is the one that maximizes the total welfare if only this type is served. At that level, marginal utility of quality, $\theta_i$, is equal to the marginal cost, $c + dq_i$, so $q_i = \frac{\theta_i - c}{d}$. 

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4 Per-Unit Royalty Rate

In this section, I study a scenario where the licensor sets a per-unit royalty rate \( t \). Solving for the SPNE by backward induction, I start by deriving the optimal strategy of the licensee in the second stage of the game. It has to choose whether to serve the whole market and pay total royalty \( t \) to the licensor (recall that the size of the market is normalized to one) or to serve only the high-type consumers and pay total royalty \((1 - \lambda)t\). If the licensee serves the whole market, it has to choose \( q_1, q_2, p_1, \) and \( p_2 \) to maximize its profit \( \pi_{2,LH} = \lambda \left( p_1 - cq_1 - d\frac{q_1^2}{2}\right) + \left( 1-\lambda \right) \left( p_2 - cq_2 - d\frac{q_2^2}{2}\right) - t.\) If the licensee serves only the high-type consumers, it has to choose \( q_2 \) and \( p_2 \) to maximize \( \pi_{2,H} = \left( 1-\lambda \right) \left( p_2 - cq_2 - d\frac{q_2^2}{2}\right) - (1-\lambda)t.\)

The following lemma specifies the condition under which the licensee chooses to serve both types of consumers.

**Lemma 1** The licensee serves both low-type and high-type consumers if \( t \leq d\frac{q_1^2}{2} \) and \( q_1 \geq 0 \), where \( q_1 = \frac{\theta_1 - c}{a} - \frac{1-\lambda}{2\lambda}(\theta_2 - \theta_1). \) Otherwise, only the high-type consumers are served.

This result stems from the standard feature of product line pricing, where the high-type consumers are left with a surplus to prevent them from switching to a low quality product. When a per-unit royalty rate is small, the licensee earns sufficient profit from the low-type consumers and chooses to offer both quality levels, serving the whole market. If a royalty rate is relatively large, it cuts into the profit margin from the low-type consumers. The licensee then decides to stop targeting this segment. The absence of a low quality

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\(^{11}\) The first subscript of \( \pi \) indicates the firm (1 is the licensor, 2 is the licensee), and the second subscript indicates the market coverage (\( LH \) is for the whole market; \( H \) is for only the high-type consumers).
alternative allows the licensee to earn a higher profit margin from the high-type consumers by extracting the whole surplus from them. The condition \( q_1 \geq 0 \) assures that the licensee serves the whole market in the baseline case with no royalty payments. Rearrange the terms in \( q_1 \) as \[
\frac{\theta_1 - c}{d} - \frac{1 - \lambda}{\lambda d} (\theta_2 - \theta_1) = \frac{\theta_2 - c}{d} - \frac{\theta_2 - \theta_1}{\lambda d}.
\] It is non-negative when
\[
\frac{\theta_2 - \theta_1}{\theta_2 - c} \leq \lambda. \tag{1}
\]

In the first stage of the game, when setting its royalty rate, the licensor has to consider how many units will be sold in the second stage. If (1) is satisfied and the licensor chooses a royalty rate equal to or lower than \( d \frac{q_2^2}{2} \) then the licensee will serve the whole market. Thus, in such scenario, the licensor should choose the highest possible rate allowing for that, i.e., \( t = d \frac{q_2^2}{2} \). Its profit will be \( d \frac{q_2^2}{2} \). Alternatively, the licensor could charge a high royalty rate that leads to the licensee serving only the high-type consumers. The licensee’s profit is then
\[
\pi_{2.H} = (1 - \lambda) \left( \hat{p}_2 -cq_2 - d \frac{q_2^2}{2} \right) - (1 - \lambda)t = (1 - \lambda)q_2 \left( \theta_2 - c - d \frac{q_2^2}{2} \right) - (1 - \lambda)t.
\]
Since \( q_2 = \frac{\theta_2 - c}{d} \), this profit is
\[
\pi_{2.H} = (1 - \lambda) d \frac{q_2^2}{2} - (1 - \lambda)t.
\]
The licensor can extract all of it by setting \( t = d \frac{q_2^2}{2} \) and earning profit \( (1 - \lambda) d \frac{q_2^2}{2} \). Comparing the licensor’s profits from the two possible per-unit royalty rates, we conclude that the licensor would choose to set a smaller rate \( t = d \frac{q_2^2}{2} \) if \( d \frac{q_2^2}{2} \geq (1 - \lambda) d \frac{q_2^2}{2} \). This inequality is equivalent to \( \frac{q_1}{q_2} \geq \sqrt{1-\lambda} \). We can express \( q_1 \) and \( q_2 \) in terms of the parameters of the model and compute the optimal strategies and profits of the firms. The results are presented in Proposition 1.

\[12\] This also would be the outcome if condition (1) is not satisfied.
Proposition 1 If the model parameters satisfy the following inequality

\[
\frac{\theta_2 - \theta_1}{\theta_2 - c} \leq \lambda(1 - \sqrt{1 - \lambda}), \tag{2}
\]

the licensor sets per-unit royalty rate \( t = \frac{(\theta_1 - c - \frac{1-\lambda}{\lambda d}(\theta_2 - \theta_1))^2}{2d} \). The licensee serves both groups of consumers, setting \( q_1 = \frac{\theta_1 - c}{d} - \frac{1-\lambda}{\lambda d} (\theta_2 - \theta_1) \), \( q_2 = \frac{\theta_2 - c}{d} \), \( p_1 = \theta_1 q_1 \), and \( p_2 = \theta_2 q_2 - (\theta_2 - \theta_1)q_1 \). The equilibrium profits are \( \pi_{1,t} = \frac{(\theta_1 - c - \frac{1-\lambda}{\lambda d}(\theta_2 - \theta_1))^2}{2d} \) and \( \pi_{2,t} = \frac{(1-\lambda)(\theta_2 - \theta_1)}{2\lambda d} \left( \frac{2(\theta_2 - c)}{d} - \frac{\theta_2 - \theta_1}{\lambda d} \right) \).

If the model parameters do not satisfy (2), the licensor sets per-unit royalty rate \( t = \frac{d(\theta_2 - c)^2}{2} \). The licensee serves only the high-type consumers, setting \( q_2 = \frac{\theta_2 - c}{d} \) and \( p_2 = \theta_2 q_2 \). The equilibrium profits are \( \pi_{1,t} = (1 - \lambda)\frac{(\theta_2 - c)^2}{2d} \) and \( \pi_{2,t} = 0 \).

One counterintuitive result from Proposition 1 is the negative relationship between the licensor’s profit and the taste for quality of the high-type consumers in the region where both groups of consumers are served. If the high-type consumers are willing to pay more for quality, the profit of the licensor decreases. This result is due to the adjustment in the optimal strategy of the licensee. As \( \theta_2 \) increases, the high-type segment becomes more attractive, and the licensee targets it with a higher price. In order to make sure these consumers do not switch to the low quality option, the licensee further deteriorates that product, decreasing \( q_1 \). Since the licensor charges \( dq_1^2/2 \), it has to offer a lower per-unit royalty rate. Thus, as long as (2) is satisfied and all consumers are served, the licensor’s profit decreases in \( \theta_2 \).

Proposition 1 illustrates that by using a high royalty rate the licensor could extract the whole surplus of the high-type consumers remaining after production costs. It would
choose not to do so and offer a lower royalty rate leading to full market coverage if the relative profitability of the high-type consumers is low. This happens when their share is small (the right-hand side of (2) is increasing in share of the low-type consumers $\lambda$), their valuation of quality $\theta_2$ is small, the valuation of quality of the low-type consumers $\theta_1$ is large, or when the cost of production is small. A comparison of (1) and (2) reveals that the region in which both groups are served shrinks under per-unit licensing. The regions with different market coverage separated by condition (2) are illustrated in Figure 1a).

Figure 1: Market Coverage under Different Types of Royalty Rates (Two Consumer Types)

The dashed line in Figure 1a) reflects condition (1) and separates the regions with different market coverage in the baseline case with no licensing. We confirm that when the difference in quality preference between the two segments is relatively high, a per-unit royalty leads to only high-type consumers being served (region I). In the portion of this region to the right of the dashed line, both groups were served in the baseline case with no
licensing. There, the high-type consumers are worse off under per-unit licensing since they are paying a higher price for the same (efficient) quality level. The low-type consumers are no longer served, but their welfare is unaffected since their surplus was fully extracted in the baseline case. If the preferences for quality are more similar between the two segments, a per-unit royalty leads to full market coverage (region II). The welfare of both consumer segments is unaffected since the licensee keeps the same prices and quality levels.

Figure 1Ba) in Online Appendix B illustrates the regions with different market coverage in the \((\lambda, c)\) space. Both groups are served when \(c\) is small and \(\lambda\) is large (region II). Not surprisingly, this is the region with the relatively high profitability of the low-type consumers: there is a large number of them and a small value of \(c\) allows for an adequate profit margin (as \(c\) increases and gets close to \(\theta_1\), the possible profit margin from the low-type consumers shrinks to zero). In the next section, I examine the effects of using ad valorem rates.

5 Ad Valorem Royalty Rate

In this section, I study a scenario where the licensor sets an ad valorem royalty rate \(v\) that is applied to the selling price of the product. If the licensee serves the whole market, it has to choose \(q_1, q_2, p_1,\) and \(p_2\) to maximize its profit \(\pi_{2,LH} = \lambda \left( p_1(1 - v) - cq_1 - d_{q1}^2 \right) + (1 - \lambda) \left( p_2(1 - v) - cq_2 - d_{q2}^2 \right)\). Standard solution techniques also apply here. The licensee extracts all surplus from the low-type consumers, setting \(p_1 = \theta_1 q_1\), and the optimal contract
leaves the high-type consumers indifferent between the two options: \( \theta_2q_2 - p_2 = \theta_2q_1 - p_1 \).

The latter equation is solved to obtain \( p_2 = \theta_2q_2 - (\theta_2 - \theta_1)q_1 \). Substitute these values into the licensee’s profit function to obtain

\[
\pi_{2,LH} = \lambda \left( \theta_1q_1(1 - v) - cq_1 - \frac{d\theta_2^2}{2} \right) + (1 - \lambda) \left( (\theta_2q_2 - (\theta_2 - \theta_1)q_1)(1 - v) - cq_2 - \frac{d\theta_2^2}{2} \right)
\]

The optimal quality levels are

\[
q_2 = \frac{\theta_2(1-v)-c}{d} \quad \text{and} \quad q_1 = \frac{\theta_1(1-v)-c}{d} - \frac{1-\lambda}{\lambda d}(\theta_2 - \theta_1)(1 - v).
\]

Since a portion of the sale price goes to the licensor, the licensee does not receive the full benefit of investing in higher quality levels. Thus, it responds by degrading the quality of both products.

The licensee chooses to serve both consumer types if \( q_1 \geq 0 \), which requires \( v \leq 1 - \frac{\lambda c}{\theta_1 - (1-\lambda)\theta_2} \) and \( \theta_1 - (1-\lambda)\theta_2 > 0 \). If \( v > 1 - \frac{\lambda c}{\theta_1 - (1-\lambda)\theta_2} \) or \( \theta_1 - (1-\lambda)\theta_2 \leq 0 \), the licensee serves only the high-type consumers by setting

\[
q_2 = \frac{\theta_2(1-v)-c}{d} \quad \text{and} \quad q_1 = \frac{\theta_1(1-v)-c}{d} - \frac{1-\lambda}{\lambda d}(\theta_2 - \theta_1)(1 - v),
\]

and charging \( \hat{p}_2 = \theta_2q_2 \). The following lemma summarizes these results.

**Lemma 2** If \( v \leq 1 - \frac{\lambda c}{\theta_1 - (1-\lambda)\theta_2} \) and \( \theta_1 - (1-\lambda)\theta_2 > 0 \), the licensee serves both types of consumers, setting

\[
q_1 = \frac{\theta_1(1-v)-c}{d} - \frac{1-\lambda}{\lambda d}(\theta_2 - \theta_1)(1 - v), \quad q_2 = \frac{(1-v)\theta_2-c}{d}
\]

and charging \( p_1 = \theta_1q_1, \quad p_2 = \theta_2q_2 - (\theta_2 - \theta_1)q_1 \). If \( v > 1 - \frac{\lambda c}{\theta_1 - (1-\lambda)\theta_2} \) or \( \theta_1 - (1-\lambda)\theta_2 \leq 0 \), the licensee serves only the high-type consumers, setting

\[
q_2 = \frac{(1-v)\theta_2-c}{d} \quad \text{and} \quad \hat{p}_2 = \theta_2q_2.
\]

Unlike per-unit royalties, in comparison to the baseline quality levels, ad valorem royalties distort the quality levels for both products downwards—the larger is the ad valorem rate, the larger is the decrease in the quality levels. Using the licensee’s best response in the second stage, it is possible to solve for the optimal strategy of the licensor in the first stage. This has to be done in several steps. First, in Lemma 3, I identify the conditions...
under which an interior local maximum of the licensor’s profit function exists in the region where only the high-type consumers are served.

**Lemma 3** If \( c > \frac{\theta_2(\theta_1-(1-\lambda)\theta_2)}{(1+\lambda)\theta_2-\theta_1} \), then the profit function of the licensor has an interior local maximum at \( v = \frac{\theta_2-c}{2\theta_2} \) in the region where only the high-type consumers are served. The licensee sets quality level \( q_2 = \frac{\theta_2-c}{2d} \) and charges price \( p_2 = \frac{\theta_2(\theta_2-c)}{2d} \). The firms’ profits are \( \pi_{1,v} = \frac{(1-\lambda)(\theta_2-c)^2}{4d} \) and \( \pi_{2,v} = \frac{(1-\lambda)(\theta_2-c)^2}{8d} \).

The restriction from Lemma 3 is a quadratic inequality in \( \theta_2 \): \( (1-\lambda)\theta_2^2 - (\theta_1 - (1 + \lambda)c)\theta_2 - c\theta_1 > 0 \). It could be solved to obtain
\[
\theta_2 > \frac{\theta_1 - (1 + \lambda)c + \sqrt{(\theta_1 - (1 + \lambda)c)^2 + 4(1 - \lambda)c\theta_1}}{2(1 - \lambda)} = \theta_2^*.
\] (3)
Thus, if the taste parameter of the high-type consumers is large enough, the licensor’s profit function has an interior local maximum in the region where only these consumers are served. The next lemma identifies the conditions under which an interior local maximum of the licensor’s profit function exists in the region where both types of consumers are served.

**Lemma 4** If \( \frac{\lambda \theta_1}{\theta_1 - (1-\lambda)\theta_2} \leq \frac{1}{2} + \frac{\lambda \theta_1}{2(\theta_1^2 - 2(1-\lambda)\theta_1 \theta_2 + (1-\lambda)\theta_2^2)} \) and \( \theta_1 - (1 - \lambda)\theta_2 > 0 \), then the profit function of the licensor has an interior local maximum at \( v = \frac{1}{2} - \frac{\lambda \theta_1}{2(\theta_1^2 - 2(1-\lambda)\theta_1 \theta_2 + (1-\lambda)\theta_2^2)} \) in the region where both types of consumers are served. The licensee sets quality levels and prices as defined in Lemma 2.

The first restriction from Lemma 4 is a cubic inequality in \( \theta_2 \). It is possible to write out a restriction on \( \theta_2 \) that satisfies it in the form of inequality \( \theta_2 \leq \bar{\theta}_2 \), but it is impossible
to work with the resulting expression of $\bar{\theta}_2$. Therefore, I checked numerically that $\bar{\theta}_2$ is always larger than $\theta_2$ defined in (3).\textsuperscript{13} Then, if $\theta_2 < \bar{\theta}_2$, the profit function of the licensor is maximized only in the region where both types of consumers are served. If $\theta_2 > \bar{\theta}_2$, the unique maximum is in the region where only the high-type consumers are served. And in the segment $[\theta_2; \bar{\theta}_2]$, the profit function has two local maxima: one in the region with both types of consumers served and one in the region with only the high-type consumers served.

Thus, it is necessary to find a critical value of $\theta_2^* \in [\theta_2; \bar{\theta}_2]$, which splits the space of quality preferences according to market coverage. I find this value numerically by directly comparing the profits of the licensor from Lemmas 3 and 4. The resulting regions are illustrated in Figure 1b). As expected, when the quality preference of the high-type consumers is sufficiently large, the licensor finds it optimal to charge a high ad valorem royalty that forces the licensee to serve only this segment (region III). It is informative to compare how the type of the royalty rate affects the size of the region where only the high-type consumers are served. From Figures 1a) and 1b), we observe that this region is larger when per-unit rates are used (region I vs. region III). This happens because under a per-unit agreement, the licensor is able to extract the largest possible surplus (at the efficient quality level) from the high-type consumers. Under an ad valorem agreement, the licensor needs to charge a high rate to force the licensee to serve only the high-type consumers. However, a high ad valorem rate leads to a larger deterioration of the optimal quality provided and decreases the surplus that could be obtained from the high-type consumers. This makes the licensor

\textsuperscript{13}These computations were performed for $c$ normalized to 1, values of $\theta_1$ between 1.1 and 2 with step of 0.01 and values of $\lambda$ between 0.01 and 0.99 with step of 0.01.
more likely to stick to low ad valorem rates that cause less quality distortions and lead to full market coverage. Therefore, in comparison to a per-unit agreement, under an ad valorem agreement, the licensor would require a larger value of $\theta_2$ in order to set the rates leading to only the high-type consumers being served.

Figure 1Bb) in Online Appendix B illustrates the regions with different market coverage in the $(\lambda, c)$ space under ad valorem rates. A comparison with Figure 1Ba) reveals that the region where only the high-type consumers are served is larger under per-unit rates (region I vs. region III). This reflects the licensor’s reluctance to use high ad valorem rates as described above. In order to avoid using such rates, the licensor accepts serving the low-type consumers when they are less profitable (characterized by a higher value of $c$ and a lower value of $\lambda$) and would not have been served under a per-unit rate (region II vs. region IV). Next, I will compare the firms’ profits under two licensing agreements. If the linear component of the cost function is absent, i.e., if the production cost of one unit of quality $q$ is $d\frac{q^2}{2}$, then this comparison could be done analytically, and the result is summarized in the following proposition.

**Proposition 2** For the cost function $C(q) = d\frac{q^2}{2}$, if $\lambda > 1/3$ and $\theta_2$ is between $\frac{(1-\lambda)(4-3\lambda)-\lambda\sqrt{9(1-\lambda)(4-3\lambda)}}{(1-\lambda)(4-7\lambda)}\theta_1$ and $\frac{\theta_1}{1-\lambda(1-\sqrt{1-\lambda})}$, then the licensee’s profit is higher under per-unit royalty rates. For all other values of model parameters, the licensee’s profit is higher under ad valorem rates. The licensor’s profit is always higher under per-unit royalty rates.

If the linear component of the cost function is present, the comparison of profits under
The qualitative findings from Proposition 2 are unchanged—the licensor always prefers per-unit royalty rates whereas
the licensee prefers per-unit royalty rates only for the intermediate difference in quality preference between the two groups and if there are enough low-type consumers. Figure 2 below illustrates the region where the licensee’s profit is higher under per-unit rates.

Figure 2: Licensee’s Preference for the Type of Royalty Rate (Two Consumer Types)

The intuition for the findings in Proposition 2 is as follows. When the difference in quality preference between the two groups, $\theta_2 - \theta_1$, is small (region III in Figure 2), the licensee offers them similar quality levels and charges similar prices. With a per-unit royalty rate that is tailored to the profit from the low-type consumers, the licensor is able to extract a large portion of the channel profit. The remaining profit is sufficiently small so that the licensee favors an ad valorem contract. Since ad valorem rates deteriorate the quality of both products and decrease the channel profit, the licensor avoids using high ad valorem rates.
rates. Thus, the licensee would get a larger profit under this arrangement.

Consider now an increase in quality preference of the high-type consumers as we move vertically up through region III in Figure 2. This increases the overall profitability of the channel. However, the profit that the licensor can extract using a per-unit rate goes down. Recall from the discussion of Proposition 1 that an increase in $\theta_2$ leads to a smaller quality level offered to the low-type consumers. Since the licensor has to tailor its royalty to this segment, it charges a smaller per-unit rate and earns a smaller profit. With an increase in the overall profit and a decrease in the profit of the licensor, the licensee’s profit grows at a high rate. Under an ad valorem agreement, an increase in $\theta_2$ leads to only a modest increases in the profits of the licensee. Eventually, the licensee’s profit under a per-unit rate overtakes its profit under an ad valorem rate. This happens at the lower bound on $\theta_2$ from Proposition 2, $\theta_2 = \frac{(1-\lambda)(4-3\lambda)-\lambda \sqrt{3(1-\lambda)(4-3\lambda)}}{(1-\lambda)(4-7\lambda)} \theta_1$. At this point, both parties prefer per-unit royalty arrangements (region II in Figure 2). As $\theta_2$ continues to increase and the licensor’s per-unit rate declines, ultimately the licensor finds it more profitable to target only the high-type consumers with a high per-unit rate. This happens at the upper bound on $\theta_2$ from Proposition 2, $\theta_2 = \frac{\theta_1}{1-\lambda(1-\lambda)}$, which is equivalent to condition (2) from Proposition 1. At this point, the licensor extracts the whole remaining surplus of the high-type consumers leaving nothing to the licensee. Thus, the licensee’s preference shifts back to ad valorem agreements (region I in Figure 2). In order for the region with both parties preferring per-unit royalty rates to exist, it is necessary to have a sufficiently large segment of low-type consumers ($\lambda > 1/3$ in Proposition 2). If there are not enough of them, the
licensor would switch its per-unit royalty rate to target only the high-type consumers before the licensee’s profit under a per-unit rate overtakes its profit under an ad valorem rate.

Figure 2B in Online Appendix B illustrates the region where the licensee’s profit is higher under per-unit rates in the $(\lambda, c)$ space. When the percent of low-valuation consumers $\lambda$ is large (region III), the licensor extracts a large portion of the channel profit by targeting these consumers with a per-unit rate. The remaining profit from a low number of high-type consumers is sufficiently small so that the licensee prefers an ad valorem rate. As the percent of low-valuation consumers declines, so does the share of the profit extracted from them. This leaves a larger profit for the licensee, who eventually prefers a per-unit rate which does not come with downward distortions of prices and qualities present in ad valorem arrangements (region II). Finally, when the percent of low-valuation consumers becomes sufficiently small, the licensee sets a per-unit rate targeting only the high-valuation consumers (region I). This leaves the licensee with no profit, and it again would rather face an ad valorem rate.

Turning to consumer welfare, the low-type consumers are either not served or have their whole surplus extracted; therefore, they have no preference for the type of the royalty rate. When both types of consumers are served, the high-type consumers prefer a per-unit royalty rate. Their surplus is $(\theta_2 - \theta_1)q_1$, and a downward distortion of the low product quality level under an ad valorem rate decreases it. In particular, the high-type consumers benefit from a per-unit rate in the region defined in Proposition 2, where both firms also prefer a per-unit royalty contract. For the higher values of $\theta_2$, we enter the region where
only the high-type consumers are served under a per-unit rate, but both types are served under an ad valorem rate. In this region, the high-type consumers prefer an ad valorem rate. Finally, for large values of $\theta_2$, only the high-type consumers are served under both royalty rates, and their surplus is fully extracted in both cases.

In summary, with a sufficient number of the low-type consumers and an intermediate difference in quality preferences between the consumer groups, all market participants prefer a per-unit contract. In other regions, the preferences of the parties are in conflict. The licensor prefers a per-unit contract whereas the licensee prefers an ad valorem contract. The high-type consumers prefer a per-unit rate if their quality preference is close to the quality preference of the low-type consumers and prefer an ad valorem rate if their quality preference is much higher.

It is important to note that the firms’ preferred types of rates when the difference in quality preferences is large (region I in Figure 2) relies on the assumption that there are only two consumer types. In this region, the licensor targets only the high-valuation consumers with a large per-unit rate, leaving the licensee with zero profit. If there are numerous consumer types with a large dispersion in their preferences, the licensor has to incorporate the loss of multiple intermediate consumer groups if it charges a per-unit rate that is too large. As it avoids charging a high per-unit rate, its preferences might shift towards an ad valorem rate. At the same time, the licensee might prefer this lower per-unit rate charged by the licensor. Indeed, as I formally show in Section 7, when there a continuum of consumer

\footnote{The low-type consumers are indifferent between the two contracts as their surplus is fully extracted either way.}
types, the licensor prefers an ad valorem rate when the quality preferences are sufficiently dispersed. Conversely, the licensee would prefer a per-unit rate under such circumstances.

A problem for the licensor using a per-unit rate is that it has to set the same rate for both quality levels. Thus, when both segments are served, the licensor targets the profit from the low-type consumers and forgoes a potentially large profit from the high-type consumers. An ad valorem rate allows the licensor to collect different revenues from the two quality levels, but the resulting downward distortion in qualities renders such arrangements undesirable. One simple adjustment of an ad valorem contract that solves the problem of quality distortion for the high quality product is a cap on the licensee’s payment for each unit it sells. This allows the licensee to improve the quality of the high quality product and to charge a large price for it without incurring extra payments to the licensor. As we will see in the next section, the licensor is able to improve on a simple ad valorem arrangement by using an ad valorem rate with a cap on the selling price, and under certain model parameters, this becomes its preferred licensing mechanism.

6 Ad Valorem Royalty Rate with a Cap on Selling Price

In this section, I study a scenario where the licensor sets an ad valorem rate $v$ with a cap on price to which this rate is applied. With a cap set at $p$, the licensee will pay at most $vp$ for each unit of a product it sells. The licensee chooses the optimal quality
levels and corresponding prices, $\bar{q}_1$, $\bar{q}_2$, $\bar{p}_1$, and $\bar{p}_2$, to maximize its profit. In order to focus on the possibility of the licensor increasing its profit in comparison to the previous strategies, we only need to consider the case of the licensee serving both segments. If the licensee ends up serving only the high-type consumers, the licensor can do no better than extracting the whole remaining surplus of this segment with a per-unit fee. Additionally, in the equilibrium, the optimal value of $\bar{p}$ should lie between $\bar{p}_1$ and $\bar{p}_2$. If $\bar{p} < \bar{p}_1$, then the outcome of the strategy is equivalent to charging a per-unit rate of $v \bar{p}$, and if $\bar{p} > \bar{p}_2$, then the outcome of the strategy is equivalent to charging an ad valorem rate of $\bar{v}$ without a cap.

The licensee’s profit is, then, $\pi_{2,LH} = \lambda \left( \bar{p}_1 (1 - \bar{v}) - c\bar{q}_1 - d\bar{q}_1^2 \right) + (1 - \lambda) \left( \bar{p}_2 - c\bar{q}_2 - d\bar{q}_2^2 \right) - (1 - \lambda) \bar{v} \bar{p}$. The last term of this formula is the licensee’s payment on sales of the high quality product, and it is convenient to label it as $T = (1 - \lambda) \bar{v} \bar{p}$. As long as $\bar{p} < \bar{p}_2$, this term is a fixed payment that does not affect the derivatives of $\pi_{2,LH}$, which are used to find a local maximum. In order to verify that the obtained solution is a global maximum, it will be necessary to confirm that the licensee does not want to switch to a strategy with both prices below $\bar{p}$.

Following the standard solution procedure, the licensee extracts all surplus from the low-type consumers, setting $\bar{p}_1 = \theta_1 \bar{q}_1$, and the optimal contract leaves the high-type consumers indifferent between the two options: $\theta_2 \bar{q}_2 - \bar{p}_2 = \theta_2 \bar{q}_1 - \bar{p}_1$. The latter equation is solved to obtain $\bar{p}_2 = \theta_2 \bar{q}_2 - (\theta_2 - \theta_1) \bar{q}_1$. Substitute these values into the licensee’s profit function to get $\pi_{2,LH} = \lambda \left( \theta_1 \bar{q}_1 (1 - \bar{v}) - c\bar{q}_1 - d\bar{q}_1^2 \right) + (1 - \lambda) \left( \theta_2 \bar{q}_2 - (\theta_2 - \theta_1) \bar{q}_1 \right) - T$. 28
The optimal quality levels are $q_2 = \frac{\theta_2 - c}{d}$ and $q_1 = \frac{\theta_1(1 - \tau) - c}{d} - \frac{1 - \lambda}{\lambda d}(\theta_2 - \theta_1)$. We note that $q_2$ is set at the socially efficient level. In comparison with an ad valorem rate without a cap, the quality of the first product is deteriorated further ($q_1 < q_1$ for the same ad valorem rate). This is done by the licensee in order to extract a larger surplus from the high-type consumers as this segment becomes more profitable when it is served a higher quality product under an ad valorem rate with a cap.

The crucial issue for the licensor is where to set the cap. For a fixed $v$, the licensor can change the amount of $T$ by varying $p$. It prefers to set $p$ at the highest possible level as long as the licensee would continue to offer quality level $q_2$ and charge price $p_2 > p$. If $p$ is set too high, the licensee will offer a lower quality level for the high-type consumers and charge a price smaller than $p$. That is, the licensee will follow the strategy from the previous section, treating $v$ as an ad valorem rate without a cap. Using the formulas from the previous section, it will set $q_1 = \frac{\theta_1(1 - \tau) - c}{d} - \frac{1 - \lambda}{\lambda d}(\theta_2 - \theta_1)(1 - \tau)$, $q_2 = \frac{\theta_2(1 - \tau) - c}{d}$ and charge $p_1 = \theta_1 q_1$, $p_2 = \theta_2 q_2 - (\theta_2 - \theta_1)q_1$. Its profit will be $\pi_{2,LH} = \lambda \left( \theta_1 q_1(1 - \tau) - cq_1 - d\frac{q_2}{2} \right) + (1 - \lambda) \left( (\theta_2 q_2 - (\theta_2 - \theta_1)q_1)(1 - \tau) - cq_2 - d\frac{q_2}{2} \right)$, which could be simplified to $\pi_{2,LH} = \lambda d\frac{q_1^2}{2} + (1 - \lambda) d\frac{q_2^2}{2}$. Similarly, the licensee’s profit under an ad valorem rate with a cap could be simplified to $\pi_{2,LH} = \lambda d\frac{q_1^2}{2} + (1 - \lambda) d\frac{q_2^2}{2} - T$. Then, for a given $\tau$, the largest payment that the licensor could get from sales of the high quality product is $T = \lambda d\frac{q_1^2 - q_2^2}{2} + (1 - \lambda) d\frac{q_2^2 - q_2^2}{2}$. The revenue it gets from the sales of the low quality product is $\lambda \tau p_1 = \lambda \tau \theta_1 q_1$. Thus, its profit function is $\pi_{1,LH} = \lambda \tau \theta_1 q_1 + \lambda d\frac{q_1^2 - q_2^2}{2} + (1 - \lambda) d\frac{q_2^2 - q_2^2}{2}$. The licensor chooses $\tau$ to maximize this profit, and the following proposition presents its optimal strategy.
Proposition 3 Under an ad valorem rate with a cap, the licensor’s optimal rate is \( v = \frac{\theta_1 (\theta_1 - \lambda c) + (1 - \lambda) \theta_2 (\theta_2 - 2 \theta_1)}{\theta_1^2 (1 + \lambda^2) + (1 - \lambda) \theta_2 (\theta_2 - 2 \theta_1)} \) and the price cap is \( \bar{p} = \frac{\theta_2 (\theta_2 (2 - \pi) - 2 c) - (\theta_2 - \theta_1) (2 \theta_1 (1 - \pi) - 2 c - \frac{1 - \lambda}{\lambda} (\theta_2 - \theta_1) (2 - \pi))}{2} \).

After deriving the optimal strategy of the licensor, it is necessary to confirm that it is consistent with the conditions that were used to set up the profit function. First, the price cap has to be set between the selling prices of the two products, i.e., \( p_1 < p < p_2 \). Second, the price cap has to set above the price of the high quality product if the licensee chooses to use \( v \) as an ad valorem rate and price both of its products under a cap, i.e., \( p_2 < \bar{p} \). Finally, the licensee should be willing to produce a low quality product, i.e., \( \bar{q}_1 > 0 \). Given the complexity of the formulas derived in Proposition 3, all of these conditions had to be checked numerically. I found that the first two conditions are always satisfied while the third condition places an upper bound on \( \theta_2 \). Table 1 gives an illustration of the outcome of setting ad valorem royalty rate with a cap for a specific set of model parameters and compares it to the outcome of the strategies from the previous sections.\(^{15}\)

We observe that in comparison to an ad valorem rate without a cap, when the licensor uses a cap, it sets a higher ad valorem rate (\( \bar{v} = 0.581 \) is greater than \( v = 0.5 \)). This forces the licensee to further deteriorate the quality of the low quality option (\( \bar{q}_1 = 0.243 \), which

\(^{15}\) The numerical computations of firms’ strategies were performed for values of \( c \) between 0 and 1 with step of 0.1, values of \( \theta_2 \) between 1.1 and 2 with step of 0.1, values of \( \theta_2 \) between \( \theta_1 + 0.1 \) and 3\( \theta_1 \) with step of 0.1 and values of \( \lambda \) between 0.1 and 0.95 with step of 0.05. The qualitative conclusions from Table 1 were preserved. The graph illustrating the strategies for \( c = 0, \lambda = 0.85, \theta_1 = 1 \) and various values of \( \theta_2 \) is provided in Figure 3B in Online Appendix B.
Table 1: Comparison of Strategies and Profits under Different Licensing Mechanisms ($c = 0; d = 1; \theta_1 = 1; \theta_2 = 2; \lambda = 0.85$)

<table>
<thead>
<tr>
<th>strategy</th>
<th>per-unit $t = 0.339$</th>
<th>ad valorem $v = 0.5$</th>
<th>ad valorem with a cap $\bar{v} = 0.581; \bar{p} = 2.545$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_1(\overline{q}_1)$</td>
<td>0.8235</td>
<td>0.412</td>
<td>0.243</td>
</tr>
<tr>
<td>$p_1(\overline{p}_1)$</td>
<td>0.8235</td>
<td>0.412</td>
<td>0.243</td>
</tr>
<tr>
<td>$q_2(\overline{q}_2)$</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$p_2(\overline{p}_2)$</td>
<td>3.177</td>
<td>1.5882</td>
<td>3.757</td>
</tr>
<tr>
<td>$\pi_1$</td>
<td>0.339</td>
<td>0.294</td>
<td>0.342</td>
</tr>
<tr>
<td>$\pi_2$</td>
<td>0.249</td>
<td>0.147</td>
<td>0.104</td>
</tr>
</tbody>
</table>

is smaller than $q_1 = 0.412$). This, in turn, allows the licensee to charge a relatively high price for its high quality product, $\overline{p}_2 = 3.757$. The licensor benefits from this high price because it is able to set a high cap ($\bar{p} = 2.545$), allowing it to collect a large profit from this segment. At the same time, the high price charged to the high-type consumers allows the licensee to collect sufficient revenue from this group, so it does not want to switch to the strategy of charging both prices below the cap. As we could observe from this table, it is possible that the licensor’s profit is higher under an ad valorem rate with a cap than under a per-unit rate. Figure 3 provides a comparison of the licensor’s profit under different types of licensing agreements for varying values of $\theta_2$.\footnote{In this figure, for the licensor’s profit function under a per-unit rate, we observe a kink at a point where it switches from a rate targeting both segments to a rate targeting only the high-type consumers. The profit function under an ad valorem rate with a cap stops at a point where $\overline{q}_1$ hits zero. Beyond this point, the licensor serves only the high-type consumers and could design its strategy to mimic a per-unit rate.}

Consistent with the analysis in the previous section, we confirm that the licensor always prefers a per-unit rate over an ad valorem rate. However, for the intermediate values of $\theta_2$, the licensor earns the highest profit by using an ad valorem rate with a cap. When $\theta_2$
is small, the low-valuation consumers are relatively profitable, and a larger profit can be obtained by a non-distortionary per-unit rate that targets the profit from that segment. When $\theta_2$ is large, the high-type consumers become very profitable, and it is more efficient to target them exclusively with a per-unit rate. For the intermediate values of $\theta_2$, if the licensor uses a per-unit rate targeting the profit from the low-type consumers, it leaves a lot of surplus to the high-type consumers. And if the licensor uses a per-unit rate targeting the high-type consumers, all of the potential profit from the low-type consumers is lost. An ad valorem rate with a cap successfully targets both segments and is the best option in this region. Figure 4 below illustrates the region in the $(\theta_1, \theta_2)$ space where the licensor’s profit is higher under an ad valorem rate with a cap (region II).

Figure 4B in Online Appendix B confirms the above intuition in the $(\lambda, c)$ space. In
order for the licensor to prefer an ad valorem rate with a cap, the profitability of the two consumer segments has to be balanced. This is achieved for the intermediate values of $\lambda$ (region II). For the licensee, the profit under an ad valorem rate with a cap is always smaller than its profit under an ad valorem rate without a cap. This is a counterintuitive finding as it seems plausible that a cap on price restricts the amount the licensor could extract from the licensee, which should benefit the latter. However, the licensor’s strategy of charging a high ad valorem rate decreases both the quality level for the low-type consumers and the portion of their price the licensee collects. A relatively high price cap keeps the licensee’s profit from the high-type consumers sufficiently low so that it does not compensate for the losses from the low-type segment. Thus, the licensee’s preferences for the royalty rate remain as shown in Figure 2. The preferences of the high-type consumers also remain
unchanged from the previous section. If both groups are served under a per-unit rate, this is the best option for them. If a per-unit rate is designed to serve only the high-type consumers, their surplus is fully extracted. Then, their preference is for an ad valorem rate without a cap. Recall that the surplus of the high-type consumers is proportional to the quality level of the low quality product. An ad valorem rate with a cap causes a larger quality distortion ($q_1 < q_1$), which leads to a smaller surplus under an ad valorem rate with a cap.

Similar to the point made in the end of the last section, the advantage of using a per-unit rate for the licensor when the quality preferences of consumers are sufficiently different, relies on the existence of only two consumer types. With multiple consumer types, as the importance of the single high-type consumer segment declines, the licensor is less likely to set a high per-unit rate targeting only that segment. It would set a lower rate, letting the licensee serve intermediate types as well. This would make a per-unit rate less appealing for the licensor and more appealing for the licensee. As I show in the next section, for a continuum of consumer types, when the quality preferences among consumers are sufficiently dispersed, the licensor would prefer an ad valorem rate with a cap while the licensee would prefer a per-unit rate.
7 Continuum of Consumer Types

In this section, I will examine the case of a continuous distribution of consumer types. Assume now that taste parameter $\theta$ is distributed on $[\theta_1, \theta_2]$ according to density $f(\theta)$ and distribution $F(\theta)$. If the licensee does not pay any royalties, the set-up is identical to the one in Mussa and Rosen (1978), and I will briefly summarize their solution. First, the optimal price charged to each type must satisfy $p(\theta) = q(\theta) - z(\theta)$, where $z(\theta) = \int_{\theta_1}^{\theta} q(s)ds$ is the surplus offered to consumer $\theta$ and $\theta$ is the lowest served consumer (equations (4) and (7) in Mussa and Rosen 1978). Second, an incremental increase in quality of the product offered to type $\theta$ will lead to an increase in revenue of $\theta f(\theta)$, but will force a firm to drop its price by the same increment for all $1 - F(\theta)$ consumers of higher type. Thus, the overall gain in revenue is $MR(\theta)f(\theta)$, where marginal revenue $MR(\theta)$ is $\theta - \frac{1-F(\theta)}{f(\theta)}$ (equation (8) in Mussa and Rosen 1978). Equating $MR(\theta)$ with marginal cost allows us to solve for an assignment of qualities $q(\theta)$. By setting $q(\theta) = 0$, we can solve for $\theta$, which is then used to find surplus function $z(\theta)$ and price schedule $p(\theta)$.

This solution works if $MR(\theta)$ is increasing everywhere, which is satisfied, for example, by all distributions with a decreasing hazard function. If $MR(\theta)$ has a decreasing portion, equating it with marginal cost leads to a decreasing assignment of qualities, which violates consumers’ incentive compatibility constraints. Then, it becomes necessary for a firm to "bunch" a group of consumers together by offering all of them the same quality and price (a technique for solving this problem in multidimensional settings is developed in Rochet and
Choné 1998). In order to obtain analytic solutions, I will work with a uniform distribution of consumer types, i.e., \( f(\theta) = \frac{1}{\theta_2 - \theta_1} \) and \( F(\theta) = \frac{\theta - \theta_1}{\theta_2 - \theta_1} \). For this distribution, \( MR(\theta) \) is increasing, which allows us to use the standard solution without bunching.

Consider, first, the scenario where the licensor sets a per-unit royalty rate \( t \). Then, the profit of the licensee is

\[
\int_{\theta_1}^{\theta_2} \left[ p(\theta) - cq(\theta) - d\frac{q(\theta)^2}{2} - t \right] f(\theta) \, d\theta.
\]

For each value of \( \theta \), we can use the assignment of qualities \( q(\theta) \) and prices \( p(\theta) \) outlined above to compute the value of the profit. The licensee will choose \( \theta \) that maximizes this profit. The following lemma outlines the condition under which the licensee chooses to serve the whole market.\(^{17}\)

**Lemma 5** If \( 2\theta_1 \geq \theta_2 + c \) and \( t \leq \frac{(2\theta_1 - \theta_2 - c)^2}{2d} \), the licensee serves the whole market. Otherwise, if \( t < \frac{(\theta_2 - c)^2}{2d} \), it serves only the consumers in \([\theta; \theta_2]\), where \( \theta = \frac{\theta_2 + c + \sqrt{2d}}{2} \), and if \( t \geq \frac{(\theta_2 - c)^2}{2d} \), no consumers are served.

Note that the value of \( t \) that separates partial and full market coverage is exactly the same as in Lemma 1 for the two-type case when \( \lambda = 0.5 \). Using the results of Lemma 5, we can solve for the licensor’s optimal strategy in the first stage. The licensor has to decide whether to choose the highest possible per-unit rate that allows the licensee to serve the whole market or to choose a higher rate that will lead to a partial market coverage. The results are summarized in the following Proposition.

**Proposition 4** If the model parameters satisfy the following inequality

\[
\frac{2\theta_1 - \theta_2 - c}{\theta_2 - c} \geq \frac{2}{3},
\]

\(^{17}\)The proof of this and the following results are in Online Appendix A.
the licensor sets per-unit royalty rate $t = \frac{(2\theta_1 - \theta_2 - c)^2}{2d}$, and the licensee serves the whole market: $\theta = \theta_1$. The licensor’s profit is $\pi_{1,t} = \frac{(2\theta_1 - \theta_2 - c)^2}{2d}$.

If the model parameters do not satisfy (4), the licensor sets per-unit royalty rate $t = \frac{2(\theta_2 - c)^2}{9d}$, and the licensee excludes some of the consumers by setting $\theta = \frac{5\theta_2 + c}{6}$. The licensor’s profit is $\pi_{1,t} = \frac{(\theta_2 - c)^3}{27d(\theta_2 - \theta_1)}$.

For both cases, the licensor’s strategy and profit depend on the lowest served consumer $\theta$ and per-unit royalty rate $t$ as follows: $q(\theta) = \frac{2\theta - \theta_2 - c}{d}$, $p(\theta) = \frac{1}{d} \left( \theta^2 + \theta \theta_2 - \theta_2 - \theta c \right)$, and

\[
\pi_{2,t} = \frac{1}{d(\theta_2 - \theta_1)} \left[ -\frac{2\theta^3}{3} + \frac{\theta_2^3}{6} + \theta^2 \theta_2 - \frac{\theta \theta_2^2}{2} + \left( \theta_2 - \theta \right) \left( -\theta c + \frac{c^2}{2} - d \cdot t \right) \right].
\]

Note that conditions (2) from Proposition 1 and (4) from this Proposition have an identical structure. Condition (2) was derived from inequality $\frac{q_1}{q_2} \geq \sqrt{1 - \lambda}$, where $q_1$ and $q_2$ are the quality levels for the two types of the consumers. The numerator in the left-hand side of (4) is the quality level offered to $\theta_1$-type consumer multiplied by $d$ whereas the denominator is the quality level offered to $\theta_2$-type consumer also multiplied by $d$. Thus, the left-hand side of (4) is also $\frac{q_1}{q_2}$.

Just like in the two-type case, in the region with full market coverage, the licensor’s profit decreases if the upper bound on consumers taste parameter, $\theta_2$, increases. With an increase in $\theta_2$, in order to prevent switching, the licensee is forced to decrease the quality level offered to lower types, $q(\theta) = \frac{2\theta - \theta_2 - c}{d}$, and corresponding prices, $p(\theta) = \frac{1}{d} \left( \theta^2 + \theta^2 - \theta \theta_2 - \theta c \right)$. As long as it is optimal for the licensor to set the rate that leads to full market coverage, $t = \frac{(2\theta_1 - \theta_2 - c)^2}{2d}$, it would have to decrease this rate. Without an increase in the number of consumers served, the licensor’s profit declines.
Figure 5a) illustrates the regions with different market coverage separated by condition (4). The dashed line separates the regions with different market coverage in the case with no licensing. The boundary between these regions is obtained by setting $q(\theta_1) = \frac{2\theta_1 - \theta_2 - c}{d}$ equal to zero. At $\theta_1 = \frac{\theta_2 + c}{2}$, condition (4) is not satisfied. In order for (4) to hold, a higher value of $\theta_1$ is required. Thus, the region with full market coverage (region II) shrinks under a per-unit licensing rate.

Figure 5: Market Coverage Under Different Types of Royalty Rates (Continuum of Consumer Types)

![Diagram](image.png)

Note. The dashed line separates the regions with different market coverage when there is no licensing.

Figure 5Ba) in Online Appendix B illustrates the regions with different market coverage in the $(\theta_1, c)$ space. All consumers are served when $\theta_1$ is large and $c$ is small (region II). This combination of model parameters leads to a relatively high profitability of the low-valuation consumers, which allows the licensee to serve them.

Now, consider the scenario where the licensor sets an ad valorem rate $v$. Then, the
licensee’s profit is \( \int \left[ (1 - v)p(\theta) - c q(\theta) - d q(\theta)^2 \right] f(\theta)d\theta \). The following lemma outlines the condition on \( v \) under which the licensee serves the whole market.

**Lemma 6** If \( 2\theta_1 \geq \theta_2 \) and \( v \leq 1 - \frac{c}{2\theta_1 - \theta_2} \), the licensee serves the whole market. Otherwise, it serves only the consumers in \([2; \theta_2]\), where \( 2 = \frac{\theta_2}{2} + \frac{c}{2(1 - \theta)} \).

Note that the cutoff value of \( v \) is exactly the same as in Lemma 2 for the two-type case when \( \lambda = 0.5 \). The next lemma identifies the condition on the model parameters under which the whole market is covered when the licensor uses an ad valorem rate.

**Lemma 7** If \( c \leq \frac{(4\theta_2^2 - 2\theta_1 \theta_2 + \theta_2^2)(2\theta_1 - \theta_2)}{2\theta_1^2 - \theta_1 \theta_2 + 2\theta_2^2} \), then the profit function of the licensor has an interior local maximum at \( v = \frac{1}{2} - \frac{3\theta_1 c}{2(4\theta_2^2 - 2\theta_1 \theta_2 + \theta_2^2)} \). The whole market is covered with the licensee using quantity assignment \( q(\theta) = \frac{(1 - v)(2\theta - \theta_2) - c}{d} \) and price assignment \( p(\theta) = \frac{(1 - v)(\theta - \theta_2 - \theta_1 c)}{d} \). The profits are \( \pi_{1,v} = \frac{v}{3d} \left( (1 - v)(4\theta_1^2 - 2\theta_1 \theta_2 + \theta_2^2) - 3\theta_1 c \right) \) and \( \pi_{2,v} = \frac{1}{d} \left[ (1 - v)^2(4\theta_1^2 - 2\theta_1 \theta_2 + \theta_2^2) - (1 - v)\theta_1 c + \frac{c^2}{2} \right] \).

The case of a partial market coverage is analyzed in part j) of Online Appendix A and requires solving a cubic equation \( (\theta_2(1 - v) - c)(2\theta_2(1 - v) + c)(1 + v) - 6v(1 - v)^2\theta_2^2 = 0 \). While a closed-form solution for the real root of this equation exists, it is impossible to work with. Thus, I find this solution numerically and then confirm whether it satisfies the condition for a partial market coverage, i.e., whether it is greater than \( 1 - \frac{c}{2\theta_1 - \theta_2} \). Figure 5b) illustrates the regions with partial and full market coverage. Similar to the two-type case, the region with a full market coverage (region IV) is larger under an ad valorem rate.
than under a per-unit rate (region II). This pattern is also present in the \((\theta_1, c)\) space as shown in Figure 5Bb) in Online Appendix B.

The comparison of firms’ profits under per-unit and ad valorem rates could be done analytically if the linear component of the cost function is absent. The findings are presented in the following proposition.

**Proposition 5** For the cost function \(C(q) = dq^2/2\), if \(\theta_1 < \frac{3+\sqrt{3}}{6}\theta_2\), the licensor’s profit is higher under an ad valorem rate, otherwise, it is higher under a per-unit rate. If \(\theta_1 < \frac{7+\sqrt{13}}{12}\theta_2\), the licensee’s profit is higher under a per-unit rate, otherwise, it is higher under an ad valorem rate.

Similarly to the findings in Proposition 2 for the two-type case, for the intermediate values of \(\theta_1\) (between \(\frac{3+\sqrt{3}}{6}\theta_2\) and \(\frac{7+\sqrt{13}}{12}\theta_2\)), both firms prefer a per-unit rate.\(^\text{18}\) One novel result for the continuous case is the licensor obtaining a higher profit under an ad valorem rate if there is sufficient variation in consumer tastes. In contrast, the licensor always preferred a per-unit rate for the two-type case (Proposition 2). With only two consumer types, when the difference between their tastes became sufficiently large, the licensor set a per-unit rate that would extract the whole surplus from the high-type consumers and obtained a larger profit than the one under an ad valorem rate. In that region, a decrease in the taste of the low-type consumers did not hurt the licensor since its profit came exclusively from the high-type consumers. With a continuous distribution of consumer

\(^\text{18}\)Since \(\frac{3+\sqrt{3}}{6}\theta_2 < \frac{7+\sqrt{13}}{12}\theta_2\), this region is non-empty.
preferences, when the tastes are sufficiently dissimilar and the market is partially covered, a per-unit rate becomes less appealing for the licensor. A lower portion of consumers is not served and a surplus is left for the higher portion. Unlike the two-type case, a decrease in $\theta_1$ decreases the profit of the licensor—even if the lowest served consumer remains the same, a larger length of segment $[\theta_1, \theta_2]$ implies that the licensor serves fewer consumers. This forces the licensor to switch its preference to an ad valorem rate when $\theta_1$ becomes sufficiently small.

The licensee, on the other hand, prefers a per-unit rate in the continuum-type case when the taste differences are large. With only two types, a per-unit rate charged by the licensor led to an exclusion of the low-type consumers and a full surplus extraction from the high-type consumers (region I in Figure 2). This left the licensee with no profit. In the continuum-type case, the licensor is unable to extract the surplus from the consumers with high preferences for the product. This surplus is left for the licensee who prefers such arrangement over an ad valorem rate that results in deterioration of quality levels and smaller prices.

Figure 6 illustrates the firms’ preferences for the case when $c$ is positive. If $c$ is greater than zero, the licensor’s preferences remain consistent with the findings in Proposition 5: a per-unit rate for the small variation in consumer tastes (region I) and an ad valorem rate otherwise (region II). The licensee’s preferences are also largely consistent with the results in Proposition 5. It prefers an ad valorem rate for the small variation in consumer tastes (region V). For the intermediate difference in consumer tastes, both firms prefer a per-unit
rate (region IV), and for the larger variation in consumer tastes, only the licensee prefers a per-unit rate (region III). The novel feature, not present in Proposition 5, is the licensee’ preference for an ad valorem rate when the difference in consumer tastes is large and $\theta_2$ is relatively small (region VI).

Figure 6: Firms’ Preferences for Different Types of Royalty Rates (Continuum of Consumer Types)

The reason for the licensee’s preference for an ad valorem rate in this region is as follows. Recall that the optimal quality level for each consumer is set at the level where the marginal cost of an incremental increase in quality, $c + dq$, is equal to the marginal revenue. From the computations in parts f) and h) in Online Appendix A, these quality levels are $\frac{2\theta_1 - \theta_2 - c}{d}$ for a per-unit rate and $\frac{(1-v)(2\theta_1 - \theta_2) - c}{d}$ for an ad valorem rate. An increase in the fixed portion of the marginal cost $c$ leads to a disproportionate drop in lower quality levels. The large dispersion in consumer tastes already leads to a partial market coverage even when $c = 0$. As $c$ increases, the licensee is forced to stop serving the consumers with
lower taste preferences, further shrinking its customer base. With the upper bound on consumer tastes $\theta_2$ also being small, this leads to a large decrease in licensee’s profits. In contrast, under an ad valorem rate, the licensor is able to cushion the effect of an increase in $c$ on quality levels by decreasing its valorem rate (recall that a per-unit rate does not affect the quality level offered to each consumer). This helps the licensee not to shrink its customer base as much, and its profits under an valorem rate overtake its profits under a per-unit rate. Figure 6B in Online Appendix B confirms that this region is also present in the $(\theta_1, c)$ space when $c$ is higher than a certain threshold value.

Finally, consider the scenario where the licensor sets an ad valorem rate $v$ with a cap $\bar{p}$. The licensee will choose the optimal level of $\theta$, which I label $\hat{\theta}$, that will be the lowest consumer for whom the price exceeds the cap, i.e., $\hat{\theta}$ is the smallest type for which $p(\hat{\theta}) \geq \bar{p}$. The optimal quantity and price schedules are still set using the rules $MR(\theta) = MC(\theta)$ and $p(\theta) = \theta q(\theta) - z(\theta)$. However, the expressions for $MR(\theta), q(\theta), z(\theta),$ and $p(\theta)$ will differ depending on whether $\theta$ is larger or smaller than $\hat{\theta}$. The licensee’s profit is

$$\int_{\hat{\theta}}^{\theta_2} \left[ (1 - v)p(\theta) - cq(\theta) - d^2\theta^2 \right] f(\theta)d\theta + \int_{\hat{\theta}}^{\theta_2} \left[ p(\theta) - cq(\theta) - d^2\theta^2 - \bar{p} \right] f(\theta)d\theta.$$  

The licensee will choose $\theta$ and $\hat{\theta}$ to maximize this profit. In the first stage, after taking into account the licensee’s response, the licensor maximizes its profit function

$$\int_{\hat{\theta}}^{\theta_2} \bar{p} f(\theta)d\theta$$

In part 1) of Online Appendix A, I show how to compute these values.

Similar to the results for the two-type case, the licensor is able to increase its profit in comparison to only charging an ad valorem rate. The licensor achieves this by increasing
its ad valorem rate, forcing the licensee to drop more consumer with lower tastes. This allows the licensee to charge higher prices for the high-type consumers, and the licensor sets a relatively high cap to extract a large profit from them. Figure 7 below illustrates the licensor’s profit under all three types of licensing arrangements.\textsuperscript{19}

We observe that an ad valorem rate with a cap is the preferred option for the licensor for a large set of model parameters. Only when the consumer tastes are relatively concentrated ($\theta_2$ is close to $\theta_1$), the licensor prefers to serve the whole market and chooses a per-unit rate that accomplishes that. Figure 8 below shows the region where the licensor prefers a per-unit rate in the $(\theta_1, \theta_2)$ space (region II).

\textsuperscript{19}The model parameter for this figure are $c = 0; \ d = 1; \ t_1 = 1$. 

44
Similar to the two-type case, when the quality preferences are concentrated, the licensor prefers a per-unit rate. As the dispersion in preferences increases, the licensor switches its preference to an ad valorem rate with a cap. A further increase in preference dispersion caused the licensor to target only the high-type segment with a per-unit rate in the two-type case and extract the whole surplus from them. This does not happen in the continuum-type case since an increase in a per-unit rate leads to a continuous reduction of the customer base. Thus, the licensor maintains its preference for an ad valorem rate with a cap. A similar pattern is observed in the \((\theta_1, c)\) space as presented in Figure 7B in Online Appendix B. For any value of \(c\), the licensor prefers a per-unit rate only when the preferences are concentrated and prefers an ad valorem rate with a cap otherwise. Just like in the two-type
case, the licensee never prefers an ad valorem rate with a cap so its preferences are either for a per-unit or an ad valorem rate without a cap as discussed above.

8 Managerial Guidelines and Concluding Remarks

The results in this paper highlight the tension that exists between licensors and licensees concerning licensing rates. I found that it is possible that the preferences of both firms are aligned with regard to the type of the licensing contract. Specifically, for intermediate dispersion in consumer preferences, both parties prefer per-unit rates. However, the region in which the preferences of the firms are aligned is relatively small. For the most part, the preferences of the parties are in conflict. For example, when the licensee can produce an extensive product line targeting a large number of distinct consumer segments (continuum-type case), it mostly prefers a per-unit contract. This is consistent with Apple’s objection to Qualcomm’s licensing practices. On the contrary, the licensor mostly prefers an ad valorem rate with a cap, which is consistent with Qualcomm’s stance.

Somewhat counterintuitively, while it seems that setting a cap on price to which an ad valorem rate is applied benefits the licensee, I find that the licensee would have been better with an ad valorem rate without this cap. The licensor is able to charge a higher ad valorem rate, which further deteriorates the quality level for the low-type consumers and extracts a larger share of surplus from them. The presence of a cap allows the licensee to set high quality levels for the high-type consumers, and the licensee extracts a large surplus from
those by setting a relatively high cap. This suggests that licensees should not automatically assume that the presence of a cap in an ad valorem agreement benefits them.

The paper’s findings bring out the possibility of using the type of the licensing contract as another dimension in negotiations about licensing rates. For example, a licensee could agree to a less preferred ad valorem with a cap arrangement, but demand a lower ad valorem rate and/or a lower cap. If multiple licensors with a similar technology compete for a contract with a licensee, they could improve their bargaining position by agreeing to use per-unit arrangements favored by licensees.

Certainly, other factors might impact the firms’ preference for one type of royalty over another. Frequently, a licensee uses several technologies patented by different firms, and, thus, has to negotiate with multiple licensors. For example, in the smartphone value chain in 2016, there were 29 licensors charging royalties totaling $14.2 billion (Galetovic et. al. 2018). Then, the preferred choice of the type of royalty payment might depend on the nature of the agreements the licensee makes with other licensors. Whereas the current paper focused on the licensee making the decision about product qualities, it is also possible to examine how different types of royalties affect the innovation effort of the licensor and its incentives to invest in the quality of the proprietary technology. Uncertainty about future demand should have a different impact on profits for different types of rates. Studying the interplay of the structure of product lines and different licensing agreement within the settings outlined above will further advance the consolidation of the two literatures that was started in this paper.
References


Appendix

Proof of Lemma 1. The licensee will serve both types of consumers if $\pi_{2, LH} \geq \pi_{2, H}$. Note that the optimal choice of quality levels and prices is not affected by the subtraction of the last term in either profit function. Thus, the optimal strategy of the licensee who serves both consumer types and pays a per-unit royalty is the same as the optimal strategy of the licensee who serves both consumer types and does not pay a royalty. This strategy (described in Section 2) is $q_1 = \frac{\theta_1 - c}{d} - \frac{1 - \lambda}{\lambda d} (\theta_2 - \theta_1)$, $p_1 = \theta_1 q_1$, $q_2 = \frac{\theta_2 - c}{d}$, $p_2 = \theta_2 q_2 - (\theta_2 - \theta_1) q_1$. Similarly, the optimal strategy of the licensee who serves only the high-type consumers and pays a per-unit royalty is the same as the optimal strategy of the licensee who serves only the high-type consumers and does not pay a royalty. This strategy is $q_2 = \frac{\theta_2 - c}{d}$ and $\hat{p}_2 = \theta_2 q_2$.

The inequality $\pi_{2, LH} \geq \pi_{2, H}$ is then equivalent to

$$
\lambda \left( p_1 - cq_1 - d q_1^2 \right) + (1 - \lambda) \left( p_2 - cq_2 - d q_2^2 \right) - t \geq (1 - \lambda) \left( \hat{p}_2 - cq_2 - d q_2^2 \right) - (1 - \lambda) t.
$$

Substitute $p_1 = \theta_1 q_1$, $p_2 = \theta_2 q_2 - (\theta_2 - \theta_1) q_1$, $\hat{p}_2 = \theta_2 q_2$ and simplify to obtain

$$
\lambda \left( \theta_1 q_1 - cq_1 - d q_1^2 \right) - (1 - \lambda)(\theta_2 - \theta_1) q_1 \geq \lambda t.
$$

This inequality is equivalent to

$$
\lambda q_1 \left( \theta_1 - c - d q_1 - \frac{1 - \lambda}{\lambda} (\theta_2 - \theta_1) \right) \geq \lambda t.
$$

Since $q_1 = \frac{\theta_1 - c}{d} - \frac{1 - \lambda}{\lambda d} (\theta_2 - \theta_1)$, the term in parentheses is $d q_1^2$ and the inequality simplifies to $\lambda d q_1^2 \geq \lambda t$, which is equivalent to $t \leq d q_1^2$. In addition, $q_1$ has to be non-negative; otherwise, only the high-type consumers are served regardless of $t$. ■
Proof of Proposition 1. It was derived in the main body of the paper that the licensor would choose a smaller rate \( t = d \frac{q_1^2}{2} \) if \( \frac{q_1}{q_2} \geq \sqrt{1 - \lambda} \). We can re-write \( q_1 \) as

\[
q_1 = \frac{\theta_1 - c}{d} - \frac{1 - \lambda}{\lambda d} (\theta_2 - \theta_1) = \frac{\lambda q_2 - (\theta_2 - \theta_1) + \lambda \theta_2 - \lambda \theta_1}{\lambda d} = \frac{\theta_2 - c}{d} - \frac{\theta_2 - \theta_1}{\lambda d}.
\]

Substitute this value for \( q_1 \) and \( q_2 = \frac{\theta_2 - c}{d} \) into inequality \( \frac{q_1}{q_2} \geq \sqrt{1 - \lambda} \) to obtain

\[
1 - \frac{\theta_2 - \theta_1}{\lambda (\theta_2 - c)} \geq \sqrt{1 - \lambda}.
\]

This is equivalent to \( \frac{\theta_2 - \theta_1}{\theta_2 - c} \leq \lambda (1 - \sqrt{1 - \lambda}) \), which is condition (2).

If it holds, condition (1), \( \frac{\theta_2 - \theta_1}{\theta_2 - c} \leq \lambda \), holds as well, so we confirm that the licensee serves both groups and \( q_1 \geq 0 \). Then, the licensor sets a per-unit royalty rate of \( t = d \frac{q_1^2}{2} = \frac{(\theta_1 - c - \frac{1 - \lambda}{\lambda} (\theta_2 - \theta_1))^2}{2} \). The licensee responds with

\[
q_1 = \frac{\theta_1 - c}{d} - \frac{1 - \lambda}{\lambda d} (\theta_2 - \theta_1),
\]

\( q_2 = \frac{\theta_2 - c}{d} \), \( p_1 = \theta_1 q_1 \), and \( p_2 = \theta_2 q_2 = (\theta_2 - \theta_1) q_1 \). The product is sold to all consumers; therefore, the licensor’s profit is

\[
\pi_{1,t} = t = \frac{(\theta_1 - c - \frac{1 - \lambda}{\lambda} (\theta_2 - \theta_1))^2}{2}.
\]

The profit of the licensee is

\[
\pi_{2,t} = \lambda \left( p_1 - cq_1 - d \frac{q_1^2}{2} \right) + (1 - \lambda) \left( p_2 - cq_2 - d \frac{q_2^2}{2} \right) - d \frac{q_2^2}{2} = \\
\lambda q_1 \left( \theta_1 - c - d \frac{q_1}{2} \right) + (1 - \lambda) \left( \theta_2 q_2 - (\theta_2 - \theta_1) q_1 - c q_2 - d \frac{q_2^2}{2} \right) - d \frac{q_2^2}{2} = \\
\lambda q_1 \left( \theta_1 - c - d \frac{q_1}{2} - \frac{1 - \lambda}{\lambda} (\theta_2 - \theta_1) \right) + (1 - \lambda) q_2 \left( \theta_2 - c - d \frac{q_2}{2} \right) - d \frac{q_2^2}{2} = \\
\lambda d \frac{q_2^2}{2} + (1 - \lambda) d \frac{q_2^2}{2} - d \frac{q_2^2}{2} = \frac{1 - \lambda}{2} d (q_2^2 - q_1^2) = \frac{1 - \lambda}{2} d (q_2 - q_1) (q_2 + q_1).
\]

Substitute \( q_1 = \frac{\theta_2 - c}{d} - \frac{\theta_2 - \theta_1}{\lambda d} \) and \( q_2 = \frac{\theta_2 - c}{d} \) to obtain

\[
\pi_{2,t} = \lambda \left( \frac{1 - \lambda}{2 \lambda} (\theta_2 - \theta_1) \right) \left( \frac{2(\theta_2 - c)}{d} - \frac{\theta_2 - \theta_1}{\lambda d} \right).
\]

When condition (2) is not satisfied, the licensor sets a per-unit royalty rate of \( t = d \frac{q_2^2}{2} = \frac{(\theta_2 - c)^2}{2d} \). The licensee sets \( q_2 = \frac{\theta_2 - c}{d} \) and \( p_2 = \theta_2 q_2 \), selling to \( 1 - \lambda \) high-type consumers. The licensee’s profit is zero, and the profit of the licensor is

\[
\pi_{1,t} = (1 - \lambda) t = (1 - \lambda) \frac{(\theta_2 - c)^2}{2d}.
\]

Proof of Lemma 3. When the licensee serves only the high-type consumers, it sets a quality level at \( q_2 = \frac{(1 - v) \theta_2 - c}{d} \) and price at \( \hat{p}_2 = \theta_2 q_2 = \theta_2 \frac{(1 - v) \theta_2 - c}{d} \). The licensor charges ad valorem rate \( v \) and sells \( 1 - \lambda \) units to the licensee who charges \( \hat{p}_2 \). Therefore, its

53
profit is \((1 - \lambda)\nu \frac{(1-v)\theta_2 - c}{d}\). Setting the derivative of this function equal to zero, we obtain
\[(1 - \lambda)\theta_2 \frac{(1-2v)\theta_2 - c}{d} = 0 . \]
Then, \(1 - 2v = \frac{c}{\theta_2}\), the solution to which is \(v = \frac{\theta_2 - c}{2\theta_2}\).

Substitute this value into the licensor’s profit function to get \(\pi_1 = (1 - \lambda) \frac{\theta_2 - c}{2\theta_2} \theta_2 \frac{\theta_2 + c}{d} = \frac{(1-\lambda)(\theta_2 - c)^2}{4d}\). The licensee’s profit is \(\pi_2 = (1 - \lambda) \left( \bar{p}_2(1 - v) - cq_2 - d\frac{q_2^2}{2} \right) = (1 - \lambda)q_2 \left( (1 - v)\theta_2 - c - \frac{(1-v)\theta_2 - c}{2} \right) = (1 - \lambda)d\frac{q_2^2}{2}\). Substitute \(q_2 = \frac{(1 - \lambda)(\theta_2 - c)}{8d}\) into the profit function to obtain \(\pi_2 = \frac{(1-\lambda)(\theta_2 - c)^2}{8d}\).

Finally, \(v\) falls into the region where only the high-type consumers are served if it satisfies the condition from Lemma 2: either \(\theta_1 - (1 - \lambda)\theta_2 \leq 0\) or \(v > 1 - \frac{\lambda c}{\theta_1 - (1 - \lambda)\theta_2}\). Consider the latter inequality in the region where \(\theta_1 - (1 - \lambda)\theta_2 > 0\). Substitute \(v = \frac{\theta_2 - c}{2\theta_2}\) to obtain \(\frac{\theta_2 - c}{2\theta_2} > \frac{\theta_1 - (1 - \lambda)\theta_2 - \lambda c}{\theta_1 - (1 - \lambda)\theta_2}\). Using \(\theta_1 - (1 - \lambda)\theta_2 > 0\), this inequality can be written as \(\theta_2 (\theta_1 - (1 - \lambda)\theta_2) - c\theta_1 + c\theta_2 - \lambda c\theta_2 > 2\theta_2 (\theta_1 - (1 - \lambda)\theta_2) - 2\lambda c\theta_2\). This simplifies to \(c((1 + \lambda)\theta_2 - \theta_1) > \theta_2 (\theta_1 - (1 - \lambda)\theta_2)\), which is equivalent to \(c > \frac{\theta_2 (\theta_1 - (1 - \lambda)\theta_2)}{(1 + \lambda)\theta_2 - \theta_1}\). Note that this inequality is also satisfied if \(\theta_1 - (1 - \lambda)\theta_2 \leq 0\) (since the right-hand side is negative), so we could keep just \(c > \frac{\theta_2 (\theta_1 - (1 - \lambda)\theta_2)}{(1 + \lambda)\theta_2 - \theta_1}\) as a condition for an interior local maximum in the region where only the high-type consumers are served. \(\blacksquare\)

**Proof of Lemma 4.** When the licensee serves both types of consumers, it sets \(p_1 = \theta_1 q_1\) and \(p_2 = \theta_2 q_2 - (\theta_2 - \theta_1)q_1\). The profit function of the licensor who sets ad valorem rate \(v\) is then \(\pi_1 = \lambda v p_1 + (1 - \lambda) v p_2 = \lambda v \theta_1 q_1 + (1 - \lambda) v (\theta_2 q_2 - (\theta_2 - \theta_1)q_1) = v (\lambda \theta_1 q_1 + (1 - \lambda)\theta_2 q_2 - (1 - \lambda)\theta_2 q_1 + (1 - \lambda)\theta_1 q_1) = v ((\theta_1 - (1 - \lambda)\theta_2)q_1 + (1 - \lambda)\theta_2 q_2)\). Substitute the quality levels set by the licensee from Lemma 2 into this profit function to obtain \(\pi_1 = v \left( (\theta_1 - (1 - \lambda)\theta_2) \left( (1 - v) \frac{\theta_1 - (1 - \lambda)\theta_2}{\lambda d} - \frac{c}{d} \right) + (1 - \lambda)\theta_2 \frac{(1-v)\theta_2 - c}{d} \right)\). Set the deriv-
tive of this profit function with respect to \( v \) equal to zero: 
\[
(1-2v)^{\frac{(\theta_1-(1-\lambda)\theta_2)^2}{\lambda d}} - \frac{c(\theta_1-(1-\lambda)\theta_2)}{d} + \frac{(1-2v)(1-\lambda)\theta_2^2}{d} - \frac{(1-\lambda)\theta_2v}{d} = 0.
\]
This simplifies to 
\[
(1-2v)\left(\frac{(\theta_1-(1-\lambda)\theta_2)^2}{\lambda} + (1-\lambda)\theta_2^2\right) = cv.
\]
This equality is equivalent to 
\[
(1-2v)\left(\theta_1^2 - 2(1-\lambda)\theta_1\theta_2 + (1-\lambda)\theta_2^2\right) = \lambda cv.
\]
Then, 
\[
v = \frac{1}{2} - \frac{\lambda c \theta_1}{2(\theta_1^2 - 2(1-\lambda)\theta_1\theta_2 + (1-\lambda)\theta_2^2)}.
\]
This value of \( v \) falls into the region where both types of consumers are served if it is less than or equal to \( 1 - \frac{\lambda c \theta_1}{\theta_1 - (1-\lambda)\theta_2} \). This inequality could be simplified to \( \frac{\lambda c \theta_1}{\theta_1 - (1-\lambda)\theta_2} \leq \frac{1}{2} + \frac{\lambda c \theta_1}{2(\theta_1^2 - 2(1-\lambda)\theta_1\theta_2 + (1-\lambda)\theta_2^2)} \). Another condition from Lemma 2 that assures that both types of consumers are served is \( \theta_1 - (1-\lambda)\theta_2 > 0 \).

**Proof of Proposition 2.** From Lemma 4, when \( c = 0 \) and \( \theta_1 - (1-\lambda)\theta_2 > 0 \), the optimal ad valorem rate is \( v = 0.5 \) and the licensee serves both types of consumers. The quality levels given in Lemma 2 are 
\[
q_1 = \frac{1}{2} \left( \frac{\theta_1}{d} - \frac{1-\lambda}{\lambda d} (\theta_2 - \theta_1) \right) = \frac{\theta_1 - (1-\lambda)\theta_2}{2\lambda d} \quad \text{and} \quad q_2 = \frac{\theta_2}{2d}.
\]
The prices, also given in Lemma 2, are 
\[
p_1 = \theta_1 q_1 \quad \text{and} \quad p_2 = \theta_2 q_2 - (\theta_2 - \theta_1) q_1.
\]
Substitute these prices into the licensor’s profit function to obtain 
\[
\pi_{1,v} = \frac{1}{2} (\lambda \theta_1 q_1 + (1-\lambda) (\theta_2 q_2 - (\theta_2 - \theta_1) q_1)) = \frac{\theta_1 q_1 + (1-\lambda) \theta_2 (q_2 - q_1)}{2}.
\]
Use the formulas for \( q_1 \) and \( q_2 \) from above to find 
\[
q_2 - q_1 = \frac{\theta_2}{2d} - \frac{\theta_1 - (1-\lambda)\theta_2}{2\lambda d} = \frac{\theta_2 - \theta_1}{2\lambda d}.
\]
Substitute this value and the expression for \( q_1 \) into the licensor’s profit function to get 
\[
\pi_{1,v} = \frac{\theta_1^2 - (1-\lambda)\theta_2 q_2 + (1-\lambda)\theta_2 (\theta_2 - \theta_1)}{2(\theta_1\theta_2 - (1-\lambda)\theta_2^2 - \theta_1^2)} = \frac{\theta_1^2 - 2(1-\lambda)\theta_1\theta_2 + (1-\lambda)\theta_2^2}{4\lambda d}.
\]
The licensee’s profit function is 
\[
\pi_{2,v} = \lambda \left( \frac{p_1}{2} - d^2 q_1^2 \right) + (1-\lambda) \left( \frac{p_2}{2} - d^2 q_2^2 \right) = \frac{1}{2} \left( \lambda (\theta_1 q_1 - dq_1^2) + (1-\lambda) (\theta_2 q_2 - (\theta_2 - \theta_1) q_1 - dq_2^2) \right) = \frac{1}{2} \left( q_1 (\lambda \theta_1 - \lambda dq_1 - (1-\lambda)(\theta_2 - \theta_1)) + q_2 (1-\lambda)(\theta_2 - dq_2) \right) = \frac{1}{2} \left( q_1 (\theta_1 - (1-\lambda)\theta_2 - \lambda dq_1 + q_2 (1-\lambda)(\theta_2 - dq_2) \right) = \frac{1}{2} \left( \frac{\theta_1 - (1-\lambda)\theta_2}{2\lambda d} \left( \theta_1 - (1-\lambda)\theta_2 - \lambda \frac{\theta_1 - (1-\lambda)\theta_2}{2\lambda} \right) + \frac{\theta_2}{2d} (1-\lambda)(\theta_2 - \frac{\theta_2}{2}) \right) = \frac{55}{55}.
\[
\frac{1}{2} \left(\frac{(\theta_1 - (1-\lambda)\theta_2)^2}{4\lambda d} + \frac{(1-\lambda)\theta_2^2}{4d}\right) + \frac{\theta_2^2 - 2(1-\lambda)\theta_1\theta_2 + (1-\lambda)^2\theta_2^2 + (\lambda - \lambda^2)\theta_2^2}{8\lambda d} = \frac{\theta_2^2 - 2(1-\lambda)\theta_1\theta_2 + (1-\lambda)\theta_2^2}{8\lambda d} = \frac{\pi_{1,v}}{2}.
\]

If \(\theta_1 - (1-\lambda)\theta_2 < 0\), which is equivalent to \(\theta_2 > \frac{\theta_1}{1-\lambda}\), the licensee serves only the high-type consumers. The profits given in Lemma 3 are \(\pi_{1,v} = \frac{(1-\lambda)\theta_2^2}{4d}\) and \(\pi_{2,v} = \frac{(1-\lambda)\theta_2^2}{8d}\).

From Proposition 1, when the licensor uses a per-unit rate, if condition (2) is satisfied, both consumer types are served. The licensor’s profit is \(\pi_{1,t} = \frac{(\theta_1 - \frac{1-\lambda}{2d} (\theta_2 - \theta_1))^2}{2\lambda^2 d}\), and the licensee’s profit is \(\pi_{2,t} = \frac{(1-\lambda)(\theta_2 - \theta_1)}{2\lambda} \left(\frac{2\theta_2^2}{d} - \frac{\theta_2 - \theta_1}{\lambda d}\right) = \frac{(1-\lambda)(\theta_2 - \theta_1)(\theta_1 - (1-\lambda)\theta_2)}{2\lambda^2 d}\).

If condition (2) is not satisfied, the licensor’s profit is \(\pi_{1,t} = (1-\lambda)\theta_2^2/2d\), and the licensee’s profit is \(\pi_{2,t} = 0\).

The region where the licensee serves only the high-type consumers under an ad valorem rate (characterized by \(\theta_2 > \frac{\theta_1}{1-\lambda(1-\sqrt{1-\lambda})}\)) is inside the region where the licensee serves only the high-type consumers under a per-unit rate (characterized in (2) of Proposition 1 as \(\theta_2 > \frac{\theta_1}{1-\lambda(1-\sqrt{1-\lambda})}\)). Therefore, there are three cases to consider: a) the licensee serves both types of consumers under both royalty types; b) the licensee serves both types of consumers under an ad valorem rate, but only the high-type consumers under a per-unit rate; and c) the licensee serves only the high-type consumers under both royalty types. I will compare the firms’ profits for each of these cases.

**Case a): The licensee serves both types of consumers under both royalty types (\(\theta_2 < \frac{\theta_1}{1-\lambda(1-\sqrt{1-\lambda})}\)).**

For this case, the licensee’s profit under a per-unit royalty rate is \(\pi_{2,t} = \frac{(1-\lambda)(\theta_2 - \theta_1)(\theta_1 - (1-2\lambda)\theta_2)}{2\lambda^2 d}\). The licensee’s profit under an ad valorem rate is \(\pi_{2,v} = \frac{\theta_2^2 - 2(1-\lambda)\theta_1\theta_2 + (1-\lambda)^2\theta_2^2}{8\lambda d}\). The licensee’s profit is higher under a per-unit rate when \(\frac{(1-\lambda)(\theta_2 - \theta_1)(\theta_1 - (1-2\lambda)\theta_2)}{2\lambda^2 d} > \frac{\theta_2^2 - 2(1-\lambda)\theta_1\theta_2 + (1-\lambda)^2\theta_2^2}{8\lambda d}\).
This inequality is equivalent to \(4(1 - \lambda)(2(1 - \lambda)\theta_1\theta_2 - \theta_1^2 - (1 - 2\lambda)\theta_2^2) > \lambda\theta_1^2 - 2\lambda(1 - \lambda)\theta_1\theta_2 + \lambda(1 - \lambda)\theta_2^2\), which simplifies to the following quadratic inequality:

\[
(1 - \lambda)(4 - 7\lambda)\theta_2^2 - 2(1 - \lambda)(4 - 3\lambda)\theta_1\theta_2 + (4 - 3\lambda)\theta_1^2 < 0. \tag{5}
\]

The discriminant of this inequality is \(4(1 - \lambda)^2(4 - 3\lambda)^2\theta_1^2 - 4(1 - \lambda)(4 - 7\lambda)(4 - 3\lambda)\theta_1^2 = 4(1 - \lambda)(4 - 3\lambda)(4 - 7\lambda + 3\lambda^2 - 4 + 7\lambda)\theta_1^2 = 12\lambda^2(1 - \lambda)(4 - 3\lambda)\theta_1^2\), which is always positive. The two roots are \(\theta_2 = \frac{2(1 - \lambda)(4 - 3\lambda)\theta_1 \pm 2\theta_1\sqrt{3(1 - \lambda)(4 - 3\lambda)}}{2(1 - \lambda)(4 - 7\lambda)} = \frac{(1 - \lambda)(4 - 3\lambda) \pm \lambda\sqrt{3(1 - \lambda)(4 - 3\lambda)}}{(1 - \lambda)(4 - 7\lambda)}\theta_1\).

If \(4 - 7\lambda < 0\), then the leading coefficient of (5) is negative; therefore this inequality is satisfied when \(\theta_2\) is either below the smaller root or above the larger root. Since \(4 - 7\lambda < 0\), the smaller root is \(\frac{(1 - \lambda)(4 - 3\lambda) + \lambda\sqrt{3(1 - \lambda)(4 - 3\lambda)}}{(1 - \lambda)(4 - 7\lambda)}\theta_1\). It is negative; therefore, \(\theta_2\) can not be below it. Hence, the only possibility is \(\theta_2 > \frac{(1 - \lambda)(4 - 3\lambda) - \lambda\sqrt{3(1 - \lambda)(4 - 3\lambda)}}{(1 - \lambda)(4 - 7\lambda)}\theta_1\).

If \(4 - 7\lambda > 0\), then the leading coefficient of (5) is positive; therefore, the inequality is satisfied when \(\theta_2\) is between the two roots, i.e. \(\frac{(1 - \lambda)(4 - 3\lambda) - \lambda\sqrt{3(1 - \lambda)(4 - 3\lambda)}}{(1 - \lambda)(4 - 7\lambda)}\theta_1 < \theta_2 < \frac{(1 - \lambda)(4 - 3\lambda) + \lambda\sqrt{3(1 - \lambda)(4 - 3\lambda)}}{(1 - \lambda)(4 - 7\lambda)}\theta_1\). In part a) of Online Appendix A, I establish that this upper bound on \(\theta_2\) is always larger than the upper bound on \(\theta_2\) that defines the region where both consumer types are served \((\frac{\theta_1}{1 - \lambda(1 - \sqrt{1 - \lambda})})\). Thus, regardless of the sign of \(4 - 7\lambda\), if \(\theta_2 > \frac{(1 - \lambda)(4 - 3\lambda) - \lambda\sqrt{3(1 - \lambda)(4 - 3\lambda)}}{(1 - \lambda)(4 - 7\lambda)}\theta_1\), the licensee’s profit is higher under a per-unit rate. In part b) of Online Appendix A, I establish that if \(\lambda > 1/3\), this lower bound on \(\theta_2\) lies within the region where both consumer types are served (i.e., \(\frac{(1 - \lambda)(4 - 3\lambda) - \lambda\sqrt{3(1 - \lambda)(4 - 3\lambda)}}{(1 - \lambda)(4 - 7\lambda)}\theta_1 < \frac{\theta_1}{1 - \lambda(1 - \sqrt{1 - \lambda})}\)). Thus, the region where the licensee prefers a per-unit rate is non-empty.

Turning to the licensor, its profit under a per-unit rate is \(\pi_{1,t} = \frac{(\theta_t - (1 - \lambda)\theta_2)^2}{2\lambda^2d}\) whereas its
profit under an ad valorem rate is \( \pi_{1,v} = \frac{\theta_1^2 - 2(1-\lambda)\theta_1\theta_2 + (1-\lambda)\theta_2^2}{4\lambda d} \). The licensor’s profit is higher under a per-unit rate when \( \frac{(\theta_1 - (1-\lambda)\theta_2)^2}{2\lambda^2 d} > \frac{\theta_1^2 - 2(1-\lambda)\theta_1\theta_2 + (1-\lambda)\theta_2^2}{4\lambda d} \). After cross-multiplication, we obtain \( 2\theta_1^2 - 4(1-\lambda)\theta_1\theta_2 + 2(1-\lambda)^2\theta_2^2 > \lambda\theta_1^2 - 2\lambda(1-\lambda)\theta_1\theta_2 + \lambda(1-\lambda)\theta_2^2 \). This inequality is equivalent to \( (2 - 5\lambda + 3\lambda^2)\theta_2^2 - (4 - 6\lambda + 2\lambda^2)\theta_1\theta_2 + (2 - \lambda)\theta_1^2 > 0 \), which simplifies to the following quadratic inequality:

\[
(1 - \lambda)(2 - 3\lambda)\theta_2^2 - 2(2 - \lambda)(1 - \lambda)\theta_1\theta_2 + (2 - \lambda)\theta_1^2 > 0.
\]

The discriminant of this inequality is \( 4(2 - \lambda)^2(1 - \lambda)^2\theta_1^2 - 4(1 - \lambda)(2 - 3\lambda)(2 - \lambda)\theta_1^2 = 4(1 - \lambda)(2 - \lambda)(2 - 3\lambda + \lambda^2 - 2 + 3\lambda)\theta_1^2 = 4\lambda^2(1 - \lambda)(2 - \lambda)\theta_1^2 \), which is always positive. The two roots are \( \theta_2 = \frac{2(2 - \lambda)(1 - \lambda)\theta_1 \pm 2\sqrt{\lambda(1 - \lambda)(2 - \lambda)}}{2(1 - \lambda)(2 - 3\lambda)} = \frac{2(2 - \lambda)(1 - \lambda)\theta_1 \pm \sqrt{(1 - \lambda)(2 - \lambda)}}{(1 - \lambda)(2 - 3\lambda)} \theta_1 \).

If \( 2 - 3\lambda < 0 \), then the leading coefficient of (6) is negative; therefore, the inequality is satisfied when \( \theta_2 \) is between the two roots, i.e. \( \frac{(2 - \lambda)(1 - \lambda) + \lambda\sqrt{(1 - \lambda)(2 - \lambda)}}{(1 - \lambda)(2 - 3\lambda)} \theta_1 < \theta_2 < \frac{(2 - \lambda)(1 - \lambda) - \lambda\sqrt{(1 - \lambda)(2 - \lambda)}}{(1 - \lambda)(2 - 3\lambda)} \theta_1 \). Since \( \frac{(2 - \lambda)(1 - \lambda) + \lambda\sqrt{(1 - \lambda)(2 - \lambda)}}{(1 - \lambda)(2 - 3\lambda)} \theta_1 < 0 \), the first inequality always holds; therefore, we require that only \( \theta_2 < \frac{(2 - \lambda)(1 - \lambda) - \lambda\sqrt{(1 - \lambda)(2 - \lambda)}}{(1 - \lambda)(2 - 3\lambda)} \theta_1 \).

If \( 2 - 3\lambda > 0 \), then the leading coefficient of (6) is positive; therefore, this inequality is satisfied when \( \theta_2 \) is either below the smaller root or above the larger root. Since \( 2 - 3\lambda > 0 \), the smaller root is \( \frac{(2 - \lambda)(1 - \lambda) - \lambda\sqrt{(1 - \lambda)(2 - \lambda)}}{(1 - \lambda)(2 - 3\lambda)} \theta_1 \). Then, just like in previous case we need \( \theta_2 < \frac{(2 - \lambda)(1 - \lambda) - \lambda\sqrt{(1 - \lambda)(2 - \lambda)}}{(1 - \lambda)(2 - 3\lambda)} \theta_1 \). In part c) of Online Appendix A, I establish that this upper bound on \( \theta_2 \) is always larger than the upper bound on \( \theta_2 \) that defines the region where both consumer types are served \( \left( \frac{\theta_1}{1 - \lambda(1 - \sqrt{1 - \lambda})} \right) \). This means that if the model parameters fall into the region where both consumer types are served under both types of royalty rates,
then \( \theta_2 < \frac{(2-\lambda)(1-\lambda)-\lambda\sqrt{(1-\lambda)(2-\lambda)}}{(1-\lambda)(2-3\lambda)} \theta_1 \) and the licensor always prefers a per-unit rate.

**Case b):** The licensee serves both types of consumers under an ad valorem rate, but only the high-type consumers under a per-unit rate \( \left( \frac{\theta_1}{1-\lambda(1-\lambda)} < \theta_2 < \frac{\theta_1}{1-\lambda} \right) \).

For this case, the licensee’s profit under a per-unit rate is zero whereas it earns a positive profit \( \pi_{2,v} = \frac{\theta_1^2-2(1-\lambda)\theta_1 \theta_2 + (1-\lambda)\theta_2^2}{8\lambda d} \) under an ad valorem rate. The licensor’s profit under a per-unit rate is \( \pi_{1,t} = (1-\lambda)\frac{\theta_2^2}{2d} \) and its profit under an ad valorem rate is \( \pi_{1,v} = \frac{\theta_1^2-2(1-\lambda)\theta_1 \theta_2 + (1-\lambda)\theta_2^2}{4\lambda d} \). The licensor prefers a per-unit rate when \( \frac{\theta_1^2-2(1-\lambda)\theta_1 \theta_2 + (1-\lambda)\theta_2^2}{4\lambda d} < (1-\lambda)\frac{\theta_2^2}{2d} \). This inequality is equivalent to \( \theta_1^2 - 2(1-\lambda)\theta_1 \theta_2 + (1-\lambda)\theta_2^2 < 2\lambda(1-\lambda)\theta_2^2 \), which simplifies to the following quadratic inequality:

\[
(1-2\lambda)(1-\lambda)\theta_2^2 - 2(1-\lambda)\theta_1 \theta_2 + \theta_1^2 < 0 \tag{7}
\]

The discriminant of this inequality is \( 4(1-\lambda)^2\theta_1^2 - 4(1-\lambda)(1-2\lambda)\theta_1^2 = 4(1-\lambda)^2\lambda \theta_1^2 \), which is always positive. The two roots are \( \theta_2 = \frac{2(1-\lambda)\theta_1 \pm \sqrt{(1-\lambda)^2 \lambda}}{2(1-2\lambda)(1-\lambda)} = \frac{1-\lambda \pm \sqrt{(1-\lambda)^2 \lambda}}{(1-2\lambda)(1-\lambda)} \theta_1 \).

If \( 1-2\lambda < 0 \), then the leading coefficient of (7) is negative; therefore, the inequality is satisfied when \( \theta_2 \) is either below the smaller root or above the larger root. Since \( 1-2\lambda < 0 \), the smaller root is \( \frac{1-\lambda - \sqrt{(1-\lambda)^2 \lambda}}{(1-2\lambda)(1-\lambda)} \theta_1 \). It is negative; therefore, \( \theta_2 \) can not be below it. Hence, the only possibility is \( \theta_2 > \frac{1-\lambda + \sqrt{(1-\lambda)^2 \lambda}}{(1-2\lambda)(1-\lambda)} \theta_1 \).

If \( 1-2\lambda > 0 \), then the leading coefficient of (7) is positive; therefore, the inequality is satisfied when \( \theta_2 \) is between the two roots, i.e. \( \frac{1-\lambda - \sqrt{(1-\lambda)^2 \lambda}}{(1-2\lambda)(1-\lambda)} \theta_1 < \theta_2 < \frac{1-\lambda + \sqrt{(1-\lambda)^2 \lambda}}{(1-2\lambda)(1-\lambda)} \theta_1 \).

In part d) of Online Appendix A, I establish that when \( 1-2\lambda > 0 \), this upper bound on \( \theta_2 \) is always larger than the upper bound on \( \theta_2 \) that defines the region for this case \( \left( \frac{\theta_1}{1-\lambda} \right) \).
This means that the second inequality is satisfied. Thus, regardless of the sign of $1 - 2\lambda$, if 
$\theta_2 > \frac{1 - \lambda - \sqrt{(1 - \lambda)^2}}{(1 - 2\lambda)(1 - \lambda)} \theta_1$, the licensee’s profit is higher under a per-unit rate. In part e) of Online Appendix A, I establish that this lower bound on $\theta_2$ is always smaller than the lower bound on $\theta_2$ that defines the region for this case ($\frac{\theta_1}{1 - \lambda(1 - \sqrt{1 - \lambda})}$). This would mean that if $\theta_2$ falls into the region for case b), i.e., $\theta_2 > \frac{\theta_1}{1 - \lambda(1 - \sqrt{1 - \lambda})}$, then $\theta_2$ is also larger than $\frac{1 - \lambda - \sqrt{(1 - \lambda)^2}}{(1 - 2\lambda)(1 - \lambda)} \theta_1$. Then, inequality (7) holds true. Thus, if $\theta_2$ falls into the region where both types of consumers are served under an ad valorem rate, but only the high-type consumers are served under a per-unit rate, then the licensor’s profit is always larger under a per-unit rate.

Case c): The licensee serves only the high-type consumers under both royalty types ($\theta_2 > \frac{\theta_1}{1 - \lambda}$).

For this case, the licensor’s profit under a per-unit rate is $\pi_{1,t} = (1 - \lambda)\frac{q_2^2}{2d}$, which is larger than the licensor’s profit under an ad valorem rate $\pi_{1,v} = \frac{(1 - \lambda)\theta_2^2}{4d}$. The licensee’s profit under a per-unit rate is zero whereas it earns a positive profit $\pi_{2,v} = \frac{(1 - \lambda)\theta_2^2}{8d}$ under an ad valorem royalty.

Proof of Proposition 3. As derived in the main body of the paper, the licensor’s profit function is $\lambda \pi \theta_1 \overline{q}_1 + \lambda q_1 \frac{\theta_1}{\theta_2} + (1 - \lambda) d \frac{q^2_1 - q^2_2}{2} + (1 - \lambda) d \frac{q^2_2 - q^2_1}{2}$. The derivative of this profit function with respect to $\overline{v}$ is $\lambda \theta_1 \overline{q}_1 + \lambda \pi \theta_1 \frac{\partial q_1}{\partial \overline{v}} + \frac{\lambda d}{2} \left(2q_1 \frac{\partial q_1}{\partial \overline{v}} - 2q_1 \frac{\partial q_1}{\partial \theta_2} \right) + \frac{(1 - \lambda)d}{2} \left(2q_2 \frac{\partial q_2}{\partial \overline{v}} - 2q_2 \frac{\partial q_2}{\partial \theta_2} \right)$. Using the formulas for the optimal quantities, $q_1 = \frac{\theta_1(1 - \overline{v}) - c}{d} - \frac{1 - \lambda}{\lambda d} (\theta_2 - \theta_1)(1 - \overline{v})$, $q_2 = \frac{\theta_2(1 - \overline{v}) - c}{d}$, $\overline{q}_1 = \frac{\theta_1(1 - \overline{v}) - c}{d} - \frac{1 - \lambda}{\lambda d} (\theta_2 - \theta_1)$, and $\overline{q}_2 = \frac{\theta_2 - c}{d}$, we can compute $\frac{\partial q_1}{\partial \overline{v}} = \frac{\theta_1}{d} + \frac{1 - \lambda}{\lambda d} (\theta_2 - \theta_1)$, $\frac{\partial q_2}{\partial \overline{v}} = -\frac{\theta_2}{d}$,
\[ \frac{\partial \eta_1}{\partial \theta} = -\frac{1}{d}, \text{ and } \frac{\partial \eta_2}{\partial \theta} = 0. \] Substituting these formulas into the derivative of the profit function and setting it equal to zero to obtain:

\[ \lambda \theta_1 \Phi_1 - \frac{\lambda \Phi_1^2}{d} + \frac{\lambda d}{2} \left( -\frac{2(\theta_1 \Phi_1)}{d_1} - 2q_1 \left( -\frac{\theta_1}{d} + \frac{1-\lambda}{d} (\theta_2 - \theta_1) \right) \right) + \frac{(1-\lambda)q_2}{2} \left( \frac{2\theta_2 q_2}{d_1} \right) = 0. \] This equation simplifies to:

\[ -\frac{\lambda \Phi_1^2}{d} + \theta_1 q_1 + (1-\lambda) \theta_2 (q_2 - q_1) = 0. \] (8)

Using the formulas for \( q_1 \) and \( q_2 \), we get:

\[ q_2 - q_1 = \frac{(\theta_2 - \theta_1)(1-\tau)}{\lambda d} \left( \frac{\theta_1^2}{2} - 1 - \frac{1-\lambda}{\lambda} (\theta_2 - \theta_1)^2 \right). \] Substituting this expression and the formula for \( q_1 \) into (8) to obtain:

\[ -\frac{\lambda \Phi_1^2}{d} + \frac{\theta_1^2}{2} (1-\tau) - \frac{c \theta_1}{d} - \frac{1-\lambda}{\lambda d} \theta_1 (\theta_2 - \theta_1) (1-\tau) + \frac{1-\lambda}{\lambda d} \theta_2 (\theta_2 - \theta_1) (1-\tau) = 0. \]

This equality simplifies to:

\[ \theta_1^2 - \frac{c \theta_1}{d} + \frac{1-\lambda}{\lambda} \theta_2 (\theta_2 - \theta_1)^2 = \tau \left( (1+\lambda) \frac{\theta_1^2}{2} + \frac{1-\lambda}{\lambda} (\theta_2 - \theta_1)^2 \right). \] The left-hand side of this equality is:

\[ \frac{\lambda \theta_1^2 - \lambda c \theta_1 + (1-\lambda) (\theta_2^2 - 2\theta_1 \theta_2 + \theta_1^2)}{\lambda} = \frac{\theta_1^2 - \lambda c \theta_1 + (1-\lambda) (\theta_2^2 - 2\theta_1 \theta_2 + \theta_1^2)}{\lambda} = \frac{\theta_1 (\theta_1 - \lambda c) + (1-\lambda) \theta_2 (\theta_2 - 2 \theta_1)}{\lambda}. \]

The right-hand side of this equality is:

\[ \tau \left( \frac{\lambda \theta_1^2 + \lambda^2 \theta_1^2 + (1-\lambda) (\theta_2^2 - 2\theta_1 \theta_2 + \theta_1^2)}{\lambda} \right) = \tau \frac{\theta_1^2 + \lambda^2 \theta_1^2 + (1-\lambda) (\theta_2^2 - 2\theta_1 \theta_2 + \theta_1^2)}{\lambda}. \]

Combining the left- and the right-hand sides together and solving for \( \tau \), we obtain:

\[ \tau = \frac{\theta_1 (\theta_1 - \lambda c) + (1-\lambda) \theta_2 (\theta_2 - 2 \theta_1)}{\theta_1^2 + \lambda^2 \theta_1^2 + (1-\lambda) \theta_2^2 (\theta_2 - 2 \theta_1)}. \]

The total payment the licensor receives from the high-type consumers is:

\[ T = (1-\lambda) \Phi_1 \Phi_2, \]

so \( \tau = \frac{1}{(1-\lambda) \Phi_1 \Phi_2} \). In the main body of the paper it was shown that:

\[ T = \lambda d \frac{\theta_1 - q_1}{2} + (1-\lambda) \frac{\theta_2^2 - q_2}{2}, \]

so:

\[ \tau \Phi = \frac{\lambda d \Phi_1 - q_1 \Phi_1 + q_1}{2(1-\lambda) \Phi_1} + \frac{d(q_2 - q_1)(\Phi_2 + q_2)}{2}. \] Using the formulas for \( \Phi_1, q_1, \) and \( q_2 \) given in the beginning of this proof, we derive:

\[ q_1 - q_1 = -\frac{1-\lambda}{\lambda d} (\theta_2 - \theta_1) \Phi_2, \]

\[ q_1 + q_1 = \frac{2 \theta_2 (1-\tau) - 2 c}{d} - \frac{1-\lambda}{\lambda d} (\theta_2 - \theta_1) (2-\tau), \]

\[ q_1 - q_2 = \frac{\tau \Phi_1}{d}, \] and:

\[ q_2 + q_2 = \frac{\theta_2 (2-\tau) - 2 c}{d}. \] Substituting these values into the formula for \( \Phi \) to obtain:

\[ \Phi = \frac{-(\theta_2 - \theta_1)(2 \theta_2 (1-\tau) - 2 c - \frac{1-\lambda}{\lambda} (\theta_2 - \theta_1) (2-\tau)) + \theta_2 (2-\tau) - 2 c}{2}. \]
Online Appendix A for Manuscript "Licensing Mechanisms for Product Lines"

a) Proof that \( \frac{(1-\lambda)(4-3\lambda)+\lambda\sqrt{3(1-\lambda)(4-3\lambda)}}{(1-\lambda)(4-7\lambda)} \theta_1 > \frac{\theta_1}{1-\lambda(1-\sqrt{1-\lambda})} \).

Recall from the main text that we need to establish this inequality only for the case when \( 4 - 7\lambda > 0 \); therefore, we can cross-multiply and keep the sign:

\[
\left(4 - 7\lambda + 3\lambda^2 + \lambda\sqrt{3(1 - \lambda)(4 - 3\lambda)}\right)\left(1 - \lambda + \lambda\sqrt{1 - \lambda}\right) > (1 - \lambda)(4 - 7\lambda),
\]

which is equivalent to

\[
(4 - 7\lambda) (1-\lambda) + (4 - 7\lambda) \lambda\sqrt{1 - \lambda} + \left(3\lambda^2 + \lambda\sqrt{3(1 - \lambda)(4 - 3\lambda)}\right)\left(1 - \lambda + \lambda\sqrt{1 - \lambda}\right) > (1 - \lambda)(4 - 7\lambda).
\]

After cancelling \((4 - 7\lambda) (1-\lambda)\) and dividing by \(\lambda\), the inequality becomes

\[
(4 - 7\lambda) \sqrt{1 - \lambda} + \left(3\lambda + \sqrt{3(1 - \lambda)(4 - 3\lambda)}\right)\left(1 - \lambda + \lambda\sqrt{1 - \lambda}\right) > 0.
\]

Since \(4 - 7\lambda > 0\), all of the terms in the left-hand side are positive and the inequality holds.
b) Proof that \( \frac{(1-\lambda)(4-3\lambda)-\lambda\sqrt{3(1-\lambda)(4-3\lambda)}}{(1-\lambda)(4-7\lambda)} \theta_1 < \frac{\theta_1}{1-\lambda(1-\sqrt{1-\lambda})} \) if \( \lambda > 1/3 \).

Assume for now that \( 4 - 7\lambda > 0 \).

After cross-multiplying, the above inequality becomes

\[
\left( 4 - 7\lambda + 3\lambda^2 - \lambda\sqrt{3(1-\lambda)(4-3\lambda)} \right) \left( 1 - \lambda + \lambda\sqrt{1-\lambda} \right) < (1 - \lambda)(4 - 7\lambda),
\]

which is equivalent to

\[
(4 - 7\lambda) (1-\lambda) + (4 - 7\lambda) \lambda\sqrt{1-\lambda} + \left( 3\lambda^2 - \lambda\sqrt{3(1-\lambda)(4-3\lambda)} \right) \left( 1 - \lambda + \lambda\sqrt{1-\lambda} \right) < (1 - \lambda)(4 - 7\lambda).
\]

After cancelling \((4 - 7\lambda)(1 - \lambda)\) and dividing by \(\lambda\), the inequality becomes

\[
(4 - 7\lambda) \sqrt{1-\lambda} + \left( 3\lambda - \sqrt{3(1-\lambda)(4-3\lambda)} \right) \left( 1 - \lambda + \lambda\sqrt{1-\lambda} \right) < 0,
\]

which is equivalent to

\[
(4 - 7\lambda) \sqrt{1-\lambda} + 3\lambda(1 - \lambda) + 3\lambda^2 \sqrt{1-\lambda} - \sqrt{3(1-\lambda)(4-3\lambda)} \left( 1 - \lambda + \lambda\sqrt{1-\lambda} \right) < 0,
\]

Combine the first and the third term and note that \(4 - 7\lambda + 3\lambda^2 = (1 - \lambda)(4 - 3\lambda)\) to obtain

\[
\sqrt{1-\lambda} (1 - \lambda) (4 - 3\lambda) + 3\lambda(1 - \lambda) - (1 - \lambda) \sqrt{3(1-\lambda)(4-3\lambda)} - \lambda(1 - \lambda) \sqrt{3(4-3\lambda)} < 0.
\]

Divide by \(1 - \lambda\) and rearrange:

\[
3\lambda + \sqrt{1-\lambda}(4 - 3\lambda) < \sqrt{3(1-\lambda)(4-3\lambda)} + \lambda\sqrt{3(4-3\lambda)}.
\]

Both parts of this inequality are positive, so we can square it and keep the sign:

\[
9\lambda^2 + (1-\lambda)(4-3\lambda)^2 + 6\lambda(4-3\lambda)\sqrt{1-\lambda} < 3(1-\lambda)(4-3\lambda) + 3\lambda^2(4-3\lambda) + 6\lambda(4-3\lambda)\sqrt{1-\lambda}.
\]

Cancel \(6\lambda(4-3\lambda)\sqrt{1-\lambda}\), move \(9\lambda^2\) to the right-hand side and \(3(1-\lambda)(4-3\lambda)\) to the left-hand side to obtain

\[
(1 - \lambda)(4 - 3\lambda)^2 - 3(1 - \lambda)(4 - 3\lambda) < 3\lambda^2(4 - 3\lambda) - 9\lambda^2,
\]
which is equivalent to

\[(4 - 3\lambda)(1 - \lambda)(1 - 3\lambda) < 3\lambda^2(1 - 3\lambda).\]

Open the first two parentheses to obtain

\[(4 - 7\lambda + 3\lambda^2)(1 - 3\lambda) < 3\lambda^2(1 - 3\lambda).\]  \hspace{1cm} (1)

Since we assumed that \(4 - 7\lambda > 0\), this inequality holds true only when \(1 - 3\lambda\) is negative, i.e. when \(\lambda > 1/3\).

Similarly, if at the start of these derivations, we assumed \(4 - 7\lambda < 0\), then the sign of the inequalities in all steps would be changed, and (1) would become \((4 - 7\lambda + 3\lambda^2)(1 - 3\lambda) > 3\lambda^2(1 - 3\lambda)\). Since now \(4 - 7\lambda + 3\lambda^2 < 3\lambda^2\), the inequality would hold if \(1 - 3\lambda < 0\), so again \(\lambda > 1/3\).
c) Proof that \[
\frac{(2-\lambda)(1-\lambda)-\lambda\sqrt{(1-\lambda)(2-\lambda)}}{(1-\lambda)(2-3\lambda)} \theta_1 > \frac{\theta_1}{1-\lambda(1-\sqrt{1-\lambda})}.
\]
Assume for now that \(2 - 3\lambda > 0\).

After cross-multiplying, the above inequality becomes
\[
(2 - 3\lambda + \lambda^2 - \lambda \sqrt{(1-\lambda)(2-\lambda)}) \left(1 - \lambda + \lambda \sqrt{1-\lambda} \right) > (1 - \lambda)(2 - 3\lambda),
\]
which is equivalent to
\[
(2 - 3\lambda)(1-\lambda) + (2 - 3\lambda)\lambda \sqrt{1-\lambda} + \left(\lambda^2 - \lambda \sqrt{(1-\lambda)(2-\lambda)} \right) \left(1 - \lambda + \lambda \sqrt{1-\lambda} \right) > (1-\lambda)(2-3\lambda).
\]

After cancelling \((2 - 3\lambda)(1 - \lambda)\) and dividing by \(\lambda\), the inequality becomes
\[
(2 - 3\lambda) \sqrt{1-\lambda} + \left(\lambda - \sqrt{(1-\lambda)(2-\lambda)} \right) \left(1 - \lambda + \lambda \sqrt{1-\lambda} \right) > 0,
\]
which is equivalent to
\[
(2 - 3\lambda) \sqrt{1-\lambda} + \lambda(1 - \lambda) + \lambda^2 \sqrt{1-\lambda} - \sqrt{(1-\lambda)(2-\lambda)} \left(1 - \lambda + \lambda \sqrt{1-\lambda} \right) > 0,
\]
Combine the first and the third term and note that \(2 - 3\lambda + \lambda^2 = (1 - \lambda)(2 - \lambda)\) to obtain
\[
\sqrt{1-\lambda}(1 - \lambda)(2 - \lambda) + \lambda(1 - \lambda) - (1 - \lambda)\sqrt{(1-\lambda)(2-\lambda)} - \lambda(1 - \lambda)\sqrt{2-\lambda} > 0.
\]
Divide by \(1 - \lambda\) and rearrange:
\[
\lambda + \sqrt{1-\lambda}(2 - \lambda) > \sqrt{(1-\lambda)(2-\lambda)} + \lambda \sqrt{2-\lambda}.
\]
Both parts of this inequality are positive, so we can square it and keep the sign:
\[
\lambda^2 + (1 - \lambda)(2 - \lambda)^2 + 2\lambda(2 - \lambda)\sqrt{1-\lambda} > (1 - \lambda)(2 - \lambda) + \lambda^2(2 - \lambda) + 2\lambda(2 - \lambda)\sqrt{1-\lambda}.
\]
Cancel \(2\lambda(2 - \lambda)\sqrt{1-\lambda}\), move \(\lambda^2\) to the right-hand side and \((1 - \lambda)(2 - \lambda)\) to the left-hand side to obtain
\[
(1 - \lambda)(2 - \lambda)^2 - (1 - \lambda)(2 - \lambda) > \lambda^2(2 - \lambda) - \lambda^2,
\]
4
which is equivalent to

$$(1 - \lambda)^2(2 - \lambda) > \lambda^2(1 - \lambda).$$

After dividing by $1 - \lambda$ and opening the parentheses, this inequality becomes $2 - 3\lambda + \lambda^2 > \lambda^2$, which is equivalent to $2 - 3\lambda > 0$. Since we assumed in the beginning of this proof that $2 - 3\lambda > 0$, this inequality holds true.

Similarly, if at the start of these derivations, we assumed $2 - 3\lambda < 0$, then the sign of the inequalities in all steps would be changed. The last inequality would become $2 - 3\lambda < 0$, which matches the assumption.
d) Proof that \( \frac{1 - \lambda + \sqrt{(1 - \lambda)\lambda}}{(1 - 2\lambda)(1 - \lambda)} > \frac{\theta_1}{1 - \lambda} \) when \( 1 - 2\lambda > 0 \).

After cross-multiplying and cancelling \( 1 - \lambda \), the above inequality becomes

\[
1 - \lambda + \sqrt{(1 - \lambda)\lambda} > 1 - 2\lambda,
\]

which is equivalent to \( \lambda + \sqrt{(1 - \lambda)\lambda} > 0 \). This inequality always holds.
e) Proof that \( \frac{1 - \lambda - \sqrt{(1 - \lambda)\lambda}}{(1 - 2\lambda)(1 - \lambda)} \cdot \theta_1 < \frac{\theta_1}{1 - \lambda(1 - \sqrt{1 - \lambda})} \).

Assume for now that \( 1 - 2\lambda > 0 \).

After cross-multiplying, the above inequality becomes

\[
(1 - \lambda)^2 + (1 - \lambda)\lambda \sqrt{1 - \lambda} - (1 - \lambda)\sqrt{(1 - \lambda)\lambda} - \lambda(1 - \lambda)\sqrt{\lambda} < (1 - 2\lambda)(1 - \lambda),
\]

which is equivalent to

\[
(1 - \lambda)(1 - \lambda - 1 + 2\lambda) + (1 - \lambda)\lambda \sqrt{1 - \lambda} < (1 - \lambda)\sqrt{(1 - \lambda)\lambda} + \lambda(1 - \lambda)\sqrt{\lambda}.
\]

After dividing by \( 1 - \lambda \), this inequality becomes

\[
\lambda + \lambda \sqrt{1 - \lambda} < \sqrt{(1 - \lambda)\lambda} + \lambda \sqrt{\lambda}.
\]

After squaring, we obtain

\[
\lambda^2 + 2\lambda^2 \sqrt{1 - \lambda} + \lambda^2(1 - \lambda) < (1 - \lambda)\lambda + 2\lambda^2 \sqrt{1 - \lambda} + \lambda^3,
\]

which is equivalent to

\[
2\lambda^2 - \lambda^3 < \lambda - \lambda^2 + \lambda^3.
\]

Divide by \( \lambda \) and simplify to get \( 2\lambda^2 - 3\lambda + 1 > 0 \). This is equivalent to \( (1 - 2\lambda)(1 - \lambda) > 0 \).

Given our original assumption that \( 1 - 2\lambda > 0 \), this inequality holds true.

Similarly, if at the start of these derivations, we assumed \( 1 - 2\lambda < 0 \), then the sign of the inequalities in all steps would be changed. The last inequality would become \( (1 - 2\lambda)(1 - \lambda) < 0 \), which would also hold true.
f) Proof of Lemma 5.

For the uniform distribution of consumer types, \( f(\theta) = \frac{1}{\theta_2 - \theta_1} \) and \( 1 - F(\theta) = \frac{\theta_2 - \theta}{\theta_2 - \theta_1} \). Then, \( MR(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)} = \theta - (\theta_2 - \theta) = 2\theta - \theta_2 \). Equating \( MR(\theta) \) with marginal cost \( c + dq(\theta) \), we obtain the optimal assignment of qualities

\[
q(\theta) = \frac{2\theta - \theta_2 - c}{d}.
\]

(2)

The surplus that has to be offered to a consumer of type \( \theta \) is \( z(\theta) = \int_0^\theta q(s)ds = \frac{1}{d} \int_0^\theta (2s - \theta_2 - c)ds \), where \( \theta \) is the lowest served consumer. After evaluating the integral, we obtain \( z(\theta) = \frac{1}{d} \left( \theta^2 - \theta_2^2 - \theta\theta_2 + \frac{\theta_2^2 - \theta\theta_2 - \theta c + \theta c}{2} \right) \). Then, the price offered to a consumer of type \( \theta \) is \( p(\theta) = \theta q(\theta) - z(\theta) \). Substituting \( q(\theta) \) from (2), \( z(\theta) \) from above, and simplifying, we obtain the optimal assignment of prices

\[
p(\theta) = \frac{1}{d} \left( \theta^2 + \theta_2^2 - \theta\theta_2 - \theta c \right).
\]

(3)

Up until this point, the analysis proceeded exactly as in Mussa and Rosen (1978). The next step is to find \( \theta \). Whereas in Mussa and Rosen (1978), \( \theta \) is found by setting \( q(\theta) \) to zero, when the licensee has to pay a royalty, it is necessary to formally set up the profit function and maximize it with respect to \( \theta \). The licensee’s profit is \( \pi(\overline{\theta}) = \int_0^{\theta_2} \left[ p(\theta) - cq(\theta) - d\frac{q(\theta)^2}{2} - t \right] f(\theta)d\theta \). Substitute \( q(\theta) \) from (2), \( p(\theta) \) from (3), and \( f(\theta) = \frac{1}{\theta_2 - \theta_1} \) to obtain \( \pi(\overline{\theta}) = \frac{1}{\theta_2 - \theta_1} \int_0^{\theta_2} \left[ \frac{\theta^2 + \theta_2^2 - \theta\theta_2 - \theta c}{d} - \frac{(2\theta - \theta_2)c - c^2}{2d} - \frac{(2\theta - \theta_2)^2 - 2(2\theta - \theta_2)c + c^2}{2d} - t \right] d\theta = \ldots \)
\[
\frac{1}{d(\theta_2 - \theta_1)} \int_{\theta}^{\theta_2} \left[ \theta^2 + \theta^2 - \theta \theta_2 - \theta \theta_2 - \theta c + c^2 - 2 \theta^2 + 2 \theta \theta_2 - \frac{\theta^2}{2} - \frac{c^2}{2} - d \cdot t \right] d\theta =
\]
\[
\frac{1}{d(\theta_2 - \theta_1)} \int_{\theta}^{\theta_2} \left[ -\theta^2 + 2 \theta \theta_2 + \theta^2 - \theta \theta_2 - \frac{\theta^2}{2} - \theta c + c^2 - 2 \theta^2 + 2 \theta \theta_2 - \frac{\theta^2}{2} + \theta c + c^2 - d \cdot t \right] d\theta =
\]
\[
\frac{1}{d(\theta_2 - \theta_1)} \left[ -\frac{\theta_3}{3} + \frac{\theta_3}{3} + \theta_2^2 + \theta_2^2 - \frac{\theta_2^3}{2} - \theta_2 c - \frac{c^2}{2} + \theta_2 - \frac{\theta_2^2}{2} + \theta_2 - \frac{\theta_2^2}{2} + \theta_2 \right] (\theta_2 - \theta) \left( -\theta c + \frac{c^2}{2} - d \cdot t \right).
\]

After simplifying, we obtain

\[
\pi(\theta) = \frac{1}{d(\theta_2 - \theta_1)} \left[ -\frac{2\theta_3}{3} + \frac{\theta_3}{3} + \theta_2^2 - \frac{\theta_2^2}{2} + \theta_2 - \frac{\theta_2^2}{2} + (\theta_2 - \theta) \left( -\theta c + \frac{c^2}{2} - d \cdot t \right) \right].
\] (4)

This profit function is a cubic function of \( \theta \) with a negative leading coefficient, therefore it is locally maximized at the largest of the two roots of the equation \( \pi'(\theta) = 0 \). The derivative of \( \pi(\theta) \) with respect to \( \theta \) is

\[
\frac{1}{d(\theta_2 - \theta_1)} \left[ -2\theta^2 + 2\theta \theta_2 - \frac{\theta^2}{2} - \theta_2 c + 2 \theta c - \frac{c^2}{2} + d \cdot t \right] =
\]
\[
\frac{1}{d(\theta_2 - \theta_1)} \left[ -2\theta^2 + 2\theta (\theta_2 + c) - \frac{(\theta_2 + c)^2}{2} + d \cdot t \right].
\]

After setting this derivative equal to zero, we obtain the following two roots of the resulting quadratic equation

\[
-\frac{2(\theta_2 + c) \pm \sqrt{4(\theta_2 + c)^2 - 4(\theta_2 + c)^2 + 8d \cdot t}}{4} = \frac{2(\theta_2 + c) \pm 2\sqrt{2d \cdot t}}{4}.
\]

Taking the largest root, we get

\[
\theta = \frac{\theta_2 + c + \sqrt{2d \cdot t}}{2}.
\] (5)

Note that if \( t = 0 \) (i.e., the licensee does not make royalty payments), \( \theta = \frac{\theta_2 + c}{2} \) is the solution to \( q(\theta) = 0 \), which is how Mussa and Rosen (1978) solve for the location of the lowest served consumer. If \( \theta = \frac{\theta_2 + c + \sqrt{2d \cdot t}}{2} \) is larger than \( \theta_1 \), then \( \theta \) is the lowest served consumer; otherwise the licensee serves the whole market and \( \theta_1 \) is the lowest served consumer.\(^1\) Inequality \( \frac{\theta_2 + c + \sqrt{2d \cdot t}}{2} > \theta_1 \) is equivalent to \( \sqrt{2d \cdot t} > 2\theta_1 - \theta_2 - c \). This inequality

\(^1\)Note that we do not need to consider the possibility that \( \theta > \theta_1 \), but the profit function is larger at \( \theta_1 \). It is indeed the case that function (4) is decreasing to the left of the smaller root of the quadratic equation,
holds if $2\theta_1 < \theta_2 + c$ or $t > \frac{(2\theta_1 - \theta_2 - c)^2}{2d}$

Finally, we need to check that the lowest served consumer is located to the left of $\theta_2$, i.e., $\frac{\theta_2 + c + \sqrt{2d^2}}{2} < \theta_2$. This is equivalent to $t < \frac{(\theta_2 - c)^2}{2d}$.

However, at those points $q(\theta) < 0$, which means that the licensee can not serve consumers in that region.
g) Proof of Proposition 4.

Consider, first, the case when $2\theta_1 \geq \theta_2 + c$, i.e., it would be possible for the licensor to set a per-unit rate that leads to a full market coverage. Using the results from Lemma 5, the licensor could choose the highest per-unit rate which allows the licensee to serve the whole market, i.e., $t^* = \frac{(2\theta_1 - \theta_2 - c)^2}{2d}$. Then, since the size of the market is normalized to one, the licensor’s profit is also $\frac{(2\theta_1 - \theta_2 - c)^2}{2d}$.

Alternatively, the licensor can choose a higher rate that leads to a partial market coverage. From (5), the lowest served consumer is $\frac{\theta + \sqrt{2d \cdot t}}{2}$. Then, the portion of consumers purchasing the product will be $1 - F(t) = \frac{\theta - \sqrt{2d \cdot t}}{\theta_2 - \theta_1} = \frac{\theta_2 - \theta_1}{2(\theta_2 - \theta_1)}$. The licensor will maximize its profit $t \frac{\theta_2 - \sqrt{2d \cdot t}}{2(\theta_2 - \theta_1)}$. It is maximized at $t = \frac{2(\theta_2 - c)^2}{9d}$. In order to fall into the partial market coverage range, this value has to be higher than $t^*$, i.e., $\frac{2(\theta_2 - c)^2}{9d} > \frac{(2\theta_1 - \theta_2 - c)^2}{2d}$. This inequality is equivalent to $\frac{2\theta_1 - \theta_2 - c}{\theta_2 - c} < \frac{2}{3}$.

Then, $\sqrt{2d \cdot t} = \frac{2}{3} (\theta_2 - c)$, and $\frac{\theta + \sqrt{2d \cdot t}}{2} = \frac{\theta_2 + c + \sqrt{2d \cdot t}}{2} = \frac{5\theta_2 + c}{6}$. The portion of consumers purchasing the product is $1 - F(t) = \frac{\theta - (5\theta_2 + c)/6}{\theta_2 - \theta_1} = \frac{\theta_2 - c}{6(\theta_2 - \theta_1)}$ and the licensor’s profit is $\frac{(\theta_2 - c)^3}{27d(\theta_2 - \theta_1)}$. Licensee’s optimal assignments of quantities and prices are given in (2) and (3), and the profit is shown in (4).

Now, consider the case when $2\theta_1 < \theta_2 + c$, in which the licensee always serves only a portion of the market regardless of the royalty rate. Then, the optimal rate and the profit of the licensor are described above. Since inequality $\frac{2\theta_1 - \theta_2 - c}{\theta_2 - c} < \frac{2}{3}$ is satisfied, the outcome in this case is covered by the previous discussion of the partial market coverage scenario.
h) Proof of Lemma 6.

When the licensee has to pay portion $v$ of the selling price to the licensor, the marginal revenue resulting from an incremental increase of quality level offered to a consumer of type $\theta$ is $MR(\theta) = (1 - v) \left( \theta - \frac{1-F(\theta)}{f(\theta)} \right) = (1 - v) \left( \theta - (\theta_2 - \theta) \right) = (1 - v)(2\theta - \theta_2)$. Equating $MR(\theta)$ with marginal cost $c + dq$, we obtain the optimal assignment of qualities

$$q(\theta) = \frac{(1 - v)(2\theta - \theta_2) - c}{d}. \quad (6)$$

The surplus that has to be offered to a consumer of type $\theta$ is $z(\theta) = \int_{\theta}^{\theta'} q(s)ds = \frac{1}{d} \int_{\theta}^{\theta'} ((1 - v)(2s - \theta_2) - c)ds$, where $\theta$ is the lowest served consumer. After evaluating the integral, we obtain $z(\theta) = \frac{1}{d} \left( (1 - v)(\theta^2 - \theta_2^2 - \theta_2(\theta_2 - \theta)) - (\theta - \theta')c \right)$. Then, the price offered to a consumer of type $\theta$ is $p(\theta) = \theta q(\theta) - z(\theta)$. Substituting $q(\theta)$ from (6), $z(\theta)$ from above, and simplifying, we obtain the optimal assignment of prices

$$p(\theta) = \frac{(1 - v)(\theta^2 + \theta_2^2 - \theta_2(\theta_2 - \theta)) - \theta c}{d}. \quad (7)$$

Substitute $q(\theta)$ from (6), $p(\theta)$ from (7), and $f(\theta) = \frac{1}{\theta_2 - \theta_1}$ into licensee’s profit function

$$\pi(\theta) = \int_{\theta}^{\theta_2} \left[ (1 - v)p(\theta) - cq(\theta) - d\frac{q(\theta)^2}{2} \right] f(\theta)d\theta$$

to obtain

$$\pi(\theta) = \frac{1}{\theta_2 - \theta_1} \int_{\theta}^{\theta_2} \left[ (1-v)^2(\theta^2 + \theta_2^2 - \theta_2(\theta_2 - \theta)) - (1-v)(2\theta - \theta_2)c - c^2 - (1-v)^2(2\theta - \theta_2)^2 - (1-v)^2(2\theta_2 - \theta_2)c + c^2 \right] d\theta = \frac{1}{d(\theta_2 - \theta_1)} \int_{\theta}^{\theta_2} \left[ (1 - v)^2(\theta^2 + \theta_2^2 - \theta_2(\theta_2 - \theta)) - (1 - v)\theta c + c^2 - (1 - v)^2(2\theta_2 - \theta_2)^2 - \theta_2^2/2 - c^2/2 \right] d\theta = \text{...}$$
\[
\frac{1}{d(\theta_2 - \theta_1)} \int_{\theta_1}^{\theta_2} \left[ -(1-v)^2 \theta^2 + 2(1-v)^2 \theta \theta_2 + (1-v)^2 \left( \theta^2 - \theta \theta_2 - \frac{\theta_2^2}{2} \right) - (1-v) \theta c + \frac{c^2}{2} \right] \, d\theta =
\]
\[
\frac{1}{d(\theta_2 - \theta_1)} \left[ (1-v)^2 \left( \frac{-\theta_2^2}{3} + \frac{\theta_2^3}{3} + \theta_2^3 - \theta^2 \theta_2 - \theta_2^2 - \theta \theta_2^2 + \frac{\theta_2^2}{2} + \theta^2 \theta_2 - \frac{\theta_2^2}{2} \right) - \right]
\]
\[
(\theta_2 - \theta) \left( (1-v) \theta c - \frac{c^2}{2} \right)
\]

After simplifying, we obtain
\[
\pi(\theta) = \frac{1}{d(\theta_2 - \theta_1)} \left[ (1-v)^2 \left( \frac{-2\theta_2^3}{3} + \frac{\theta_2^3}{6} + \theta^2 \theta_2 - \frac{\theta_2^2}{2} \right) - (\theta_2 - \theta) \left( (1-v) \theta c - \frac{c^2}{2} \right) \right].
\]

(8)

The derivative of \(\pi(\theta)\) with respect to \(\theta\) is
\[
\frac{1}{d(\theta_2 - \theta_1)} \left[ (1-v)^2 \left( -2\theta_2^2 + 2\theta \theta_2 - \frac{\theta_2^2}{2} \right) + 2(1-v) \theta c - \frac{c^2}{2} - (1-v) \theta_2 c \right] =
\]
\[
\frac{1}{d(\theta_2 - \theta_1)} \left[ -2(1-v)^2 \theta_2^2 + 2(1-v)((1-v) \theta_2 + c) \theta - (1-v)^2 \frac{\theta_2^2}{2} - \frac{c^2}{2} - (1-v) \theta_2 c \right] =
\]
\[
\frac{1}{d(\theta_2 - \theta_1)} \left[ -2(1-v)^2 \theta_2^2 + 2(1-v)((1-v) \theta_2 + c) \theta - \frac{(1-v) \theta_2 + c)^2}{2} \right] = \frac{-2(1-v)^2((1-v) \theta_2 + c)^2}{2d(\theta_2 - \theta_1)} \leq 0.
\]

Since the derivative of \(\pi(\theta)\) is always non-positive, the profit function is decreasing in \(\theta\).

Thus, the licensee will choose the lowest possible \(\theta\) that results in a non-negative quantity.

The solution to \(q(\theta) = \frac{(1-v)(2\theta_2 - \theta_2) - c}{d} = 0\) is
\[
\theta = \frac{\theta_2}{2} + \frac{c}{2(1-v)}.
\]

(9)

If it is larger than \(\theta_1\), then \(\theta\) is the lowest served consumer; otherwise the licensee serves the whole market and \(\theta_1\) is the lowest served consumer. Inequality \(\frac{\theta_2}{2} + \frac{c}{2(1-v)} > \theta_1\) is equivalent to \(\frac{c}{1-v} > 2\theta_1 - \theta_2\). This inequality holds if \(2\theta_1 < \theta_2\) or \(v > 1 - \frac{c}{2\theta_1 - \theta_2}\).
i) Proof of Lemma 7.

Assume that the market is fully covered. After finding the optimal ad valorem rate, we will derive the condition under which this rate results in full market coverage. Use the optimal price assignment from (7) and \( \theta = \theta_1 \) to set up the licensor’s profit function:

\[
\pi_{1,v} = \frac{\theta_2}{\theta_1} \int_{\theta_1}^{\theta_2} \left( 1 - v \right) \left( \frac{\theta_1^3}{3} - \frac{\theta_1^3}{3} + \theta_1^2 \theta_2 - \theta_1 \theta_2^2 + \frac{\theta_1^2 \theta_2}{2} \right) - (\theta_2 - \theta_1) \theta_1 c \, d\theta.
\]

Evaluate the integral to obtain

\[
\pi_{1,v} = \frac{1}{d(\theta_2-\theta_1)} \left( (1 - v) \left( \frac{\theta_2^3}{3} - \frac{\theta_2^3}{3} + 2\theta_1^2 \theta_2 - \theta_1^2 \theta_2^2 - (\theta_2 - \theta_1) \theta_1 c \right) = \frac{1}{d(\theta_2-\theta_1)} \left( (1 - v) \left( \frac{\theta_2^3}{3} - \frac{2\theta_1^3}{3} + 2\theta_1^2 \theta_2 - \theta_1^2 \theta_2^2 \right) - (\theta_2 - \theta_1) \theta_1 c \right) = \frac{1}{d(\theta_2-\theta_1)} \left( (1 - v) \left( \frac{\theta_2^3}{3} - \frac{2\theta_1^3}{3} + 2\theta_1^2 \theta_2 - \theta_1^2 \theta_2^2 \right) - (\theta_2 - \theta_1) \theta_1 c \right) = \frac{v}{3d} \left( (1 - v) \left( 4\theta_1^2 - 2\theta_1 \theta_2 + \theta_2^2 \right) - 3\theta_1 c \right).
\]

This profit function is maximized when \( v = \frac{1}{2} - \frac{3\theta_1 c}{2(4\theta_1^2 - 2\theta_1 \theta_2 + \theta_2^2)} \). In order for the market to be fully covered, this rate has to be at most \( 1 - \frac{c}{2\theta_1 - \theta_2} \) (Lemma 6). Inequality \( \frac{1}{2} - \frac{3\theta_1 c}{2(4\theta_1^2 - 2\theta_1 \theta_2 + \theta_2^2)} \leq 1 - \frac{c}{2\theta_1 - \theta_2} \) is equivalent to \( c \leq \frac{6\theta_1^2 + 3\theta_1 \theta_2 + 8\theta_2^2 - 4\theta_1 \theta_2 + 2\theta_2^2}{2(4\theta_1^2 - 2\theta_1 \theta_2 + \theta_2^2)} \), which simplifies to

\[
c \leq \frac{(4\theta_1^2 - 2\theta_1 \theta_2 + \theta_2^2)(2\theta_1 - \theta_2)}{2\theta_1^2 - \theta_1 \theta_2 + 2\theta_2^2}.
\]

If this condition is satisfied, the licensor sets the rate \( v = \frac{1}{2} - \frac{3\theta_1 c}{2(4\theta_1^2 - 2\theta_1 \theta_2 + \theta_2^2)} \) and the licensee responds with quality assignment from (6) \( q(\theta) = \frac{(1-v)(2\theta - \theta_2) - c}{d} \). The licensee’s price assignment is \( p(\theta) = \frac{(1-v)(\theta^2 + \theta_2^2 - \theta_1 \theta_2) - \theta_1 c}{d} \). Finally, we could substitute \( \theta = \theta_1 \) into (8) to obtain

\[
\pi(\theta) = \frac{1}{d(\theta_2-\theta_1)} \left[ (1 - v)^2 \left( \frac{\theta_1^3}{3} + \theta_1^2 \theta_2 - \theta_1 \theta_2^2 - \theta_1^2 \theta_2^2 \right) - (\theta_2 - \theta_1) \left( (1 - v) \theta_1 c - \frac{\theta_1^2}{2} \right) \right] = \frac{1}{d(\theta_2-\theta_1)} \left[ (1 - v)^2 \left( \frac{\theta_1^3}{3} + \theta_1^2 \theta_2 - \theta_1 \theta_2^2 - \theta_1^2 \theta_2^2 \right) - (\theta_2 - \theta_1) \left( (1 - v) \theta_1 c - \frac{\theta_1^2}{2} \right) \right] = \frac{1}{d} \left[ (1 - v)^2 \left( \frac{4\theta_1^2}{6} - 2\theta_1 \theta_2 + \theta_2^2 \right) - (1 - v) \theta_1 c + \frac{\theta_1^2}{2} \right].
\]
j) Derivation of the optimal ad valorem rate under partial market coverage.

If the market is partially covered, use the optimal price assignment from (7) to set up the licensor’s profit function: \( \pi_{1,v} = \int_0^{\theta_2} v \left( \frac{(1-v)(\theta^2 + \theta^2 - \theta) - \theta c}{d} \right) \frac{1}{\theta_2 - \theta_1} \, d\theta. \) Evaluate the integral to obtain

\[
\pi_{1,v} = \frac{1}{d(\theta_2 - \theta_1)} v \left( 1 - v \right) \left( \frac{\theta_2}{3} - \frac{\theta_3}{3} + \theta^2 \theta_2 - \theta^3 - \theta^2 \theta_2 + \theta^2 \theta_2 \right) - (\theta_2 - \theta) \theta c \]

\[
= \frac{1}{d(\theta_2 - \theta_1)} v \left( 1 - v \right) \left( \frac{\theta_2}{3} - \frac{\theta_2}{2} \right) - (\theta_2 - \theta) \theta c \]

\[
= \frac{1}{d(\theta_2 - \theta_1)} v \left( 1 - v \right) \frac{(\theta_2 - \theta)(4\theta_2^2 - 2\theta_2 + \theta_2^2)}{3} - (\theta_2 - \theta) \theta c \]

\[
= \frac{\theta_2 - \theta}{3d(\theta_2 - \theta_1)} v \left( 1 - v \right) \left( 4\theta_2^2 - 2\theta_2 + \theta_2^2 - 3\theta c \right). \]

Use \( \theta = \frac{\theta_2}{2} + \frac{c}{2(1-v)} \) from (9) to compute \( \theta_2 - \theta = \frac{\theta_2(1-v) - c}{2(1-v)} \) and \( 4\theta_2^2 = \frac{\theta_2(1-v)^2 + 2\theta_2c(1-v) + c^2}{(1-v)^2}. \)

Substitute these values in the licensor’s profit function to obtain

\[
\pi_{1,v} = \frac{(\theta_2(1-v)-c)}{6d(\theta_2 - \theta_1)(1-v)} v \left( 1 - v \right) \frac{\theta_2(1-v)^2 + 2\theta_2c(1-v) + c^2 - \theta_2(1-v)^2 - \theta_2c(1-v) + \theta_2^2(1-v)^2}{(1-v)^2} - \frac{3\theta_2c(1-v) + 3c^2}{2(1-v)} \]

\[
= \frac{(\theta_2(1-v)-c)}{12d(\theta_2 - \theta_1)(1-v)^2} v \left( 1 - v \right) \frac{(\theta_2(1-v)-c)^2(2\theta_2(1-v) + c) - 2v\theta_2(\theta_2(1-v) - c)(2\theta_2(1-v) + c) - 2v\theta_2(\theta_2(1-v) - c)^2)(1-v)^2 + 2(1-v)v(\theta_2(1-v) - c)^2(2\theta_2(1-v) + c)}{(1-v)^2} \]

\[
= \frac{(\theta_2(1-v)-c)}{12d(\theta_2 - \theta_1)(1-v)^2} v \left( 1 - v \right) \frac{2v\theta_2(\theta_2(1-v) - c)(2\theta_2(1-v) + c) - 2v\theta_2(\theta_2(1-v) + c)}{(1-v)^2} \]

\[
= \frac{(\theta_2(1-v)-c)}{12d(\theta_2 - \theta_1)(1-v)^2} v \left( 1 - v \right) \frac{[(\theta_2(1-v) - c)(2\theta_2(1-v) + c) - 6v\theta_2(1-v)](1-v) + 2v(\theta_2(1-v) - c)(2\theta_2(1-v) + c)]}{(1-v)^2} \]

\[
= \frac{(\theta_2(1-v)-c)}{12d(\theta_2 - \theta_1)(1-v)^2} v \left( 1 - v \right) \frac{(\theta_2(1-v) - c)(2\theta_2(1-v) + c)(1+v) - 6v(1-v)^2\theta_2^2}{(1-v)^2}. \]

This derivative is equal to zero when \( \theta_2(1-v) - c = 0 \), but at that point, the value of the licensor’s profit function is zero and it is a local minimum. The local maximum of the profit function occurs at \( v \) which solves a cubic equation \( (\theta_2(1-v) - c)(2\theta_2(1-v) + c)(1+v) - 6v(1-v)^2\theta_2^2 = 0 \).
\( v) - 6v(1 - v)^2 \theta_2^2 = 0 \). A closed-form expression of the real root of this equation exists, but it is impossible to work with. Therefore, it is necessary to find the solution numerically and then check that it satisfies the partial market coverage condition \( v > 1 - \frac{c}{2 \theta_1 - \theta_2} \).
k) Proof of Proposition 5.

First, consider the per-unit royalty rate scenario. From Proposition 4, when \( c = 0 \), if \( \frac{2\theta_1 - \theta_2}{\theta_2} \geq \frac{2}{3} \) (which is equivalent to \( \theta_1 \geq \frac{5\theta_2}{6} \)), there is full market coverage. The profit of the licensor is \( \pi_{1,t} = \frac{(2\theta_1 - \theta_2)^2}{2d} \). Substitute \( \theta = \theta_1 \) and \( t = \frac{(2\theta_1 - \theta_2)^2}{2d} \) into the licensee’s profit to obtain

\[
\pi_{2,t} = \frac{1}{d(\theta_2 - \theta_1)} \left[ -\frac{\theta_1^3}{3} + \frac{\theta_1^2}{2} + \theta_1 \theta_2 - \theta_1 \frac{\theta_2^2}{2} - (\theta_2 - \theta_1) \frac{(2\theta_1 - \theta_2)^2}{2d} \right] =
\]

\[
\frac{1}{d} \left[ \frac{4\theta_1^3 + 2\theta_1^2 - 3\theta_1 \theta_2}{6} - (\theta_2 - \theta_1) \frac{(2\theta_1 - \theta_2)^2}{2} \right] =
\]

\[
\frac{1}{d} \frac{4\theta_1^3 + 2\theta_1^2 - 3\theta_1 \theta_2}{6}
\]

If \( \frac{2\theta_1 - \theta_2}{\theta_2} < \frac{2}{3} \) (which is equivalent to \( \theta_1 < \frac{5\theta_2}{6} \)), there is partial market coverage. The profit of the licensor is \( \pi_{1,t} = \frac{\theta_1^3}{2d(\theta_2 - \theta_1)} \). Substitute \( \theta = \frac{5\theta_2}{6} \) and \( t = \frac{2\theta_2^2}{3d} \) into the licensee’s profit to obtain

\[
\pi_{2,t} = \frac{1}{d(\theta_2 - \theta_1)} \left[ -\frac{\theta_1^3}{3} + \frac{\theta_1^2}{6} + \frac{\theta_1 \theta_2}{6} \right] = \frac{1}{d(\theta_2 - \theta_1)} \left[ \frac{25\theta_1^3}{12} - \frac{\theta_1 \theta_2}{12} - \frac{\theta_2^2}{12} \right] =
\]

\[
\frac{1}{d(\theta_2 - \theta_1)} \frac{\theta_1^3}{324}
\]

Now, consider the ad valorem rate scenario. The condition \( v \leq 1 - \frac{c}{2\theta_1 - \theta_2} \) from Lemma 7 is always satisfied when \( c = 0 \). Therefore, the boundary of the region separating full and partial market coverage is given by the second condition from this Lemma. If \( 2\theta_1 \geq \theta_2 \) (or \( \theta_1 \geq \frac{\theta_2}{2} \)), the market is fully covered. From Lemma 7, the licensor’s rate is \( v = 0.5 \) and its profit is \( \pi_{1,v} = 0.5 \left( 1 - 0.5 \right) \left( 4\theta_1^2 - 2\theta_1 \theta_2 + \theta_2^2 \right) = 4\theta_1^2 - 2\theta_1 \theta_2 + \theta_2^2 \). The profit of the licensee is

\[
\pi_{2,v} = \frac{1}{d} \left( 1 - 0.5 \right) \left( 4\theta_1^2 - 2\theta_1 \theta_2 + \theta_2^2 \right) = \frac{4\theta_1^2 - 2\theta_1 \theta_2 + \theta_2^2}{2d}.
\]

If \( 2\theta_1 \geq \theta_2 \) (or \( \theta_1 < \frac{\theta_2}{2} \)), there is partial market coverage. From part j) above, when \( c = 0 \), the cubic equation that needs to be solved for the optimal ad valorem rate is
\[(\theta_2(1-v))(2\theta_2(1-v))(1+v) - 6v(1-v)^2\theta_2^2 = 0.\] It is equivalent to \[2(1+v) - 6v = 0,\] the solution to which is \(v = 0.5.\) Substitute this value into the licensor’s profit function (also from part j) above to obtain \(\pi_{1,v} = \frac{1}{12d(\theta_2-\theta_1)} \frac{0.5(0.5\theta_2)^2(1-0.5)}{(1-0.5)^2} = \frac{\theta_2^3}{24d(\theta_2-\theta_1)}.\) Then, from (9), \(\theta = \theta_2/2.\) Substitute it into licensee’s profit from (8) to obtain \(\pi_{2,v} = \frac{1}{d(\theta_2-\theta_1)}(1-0.5)^2 \left(-\frac{\theta_2^4}{12} + \frac{\theta_2^3}{6} + \frac{\theta_2^4}{4} - \frac{\theta_2^4}{4}\right) = \frac{1}{4d(\theta_2-\theta_1)} \frac{-\theta_2^4+2\theta_2^3}{12} = \frac{\theta_2^3}{48d(\theta_2-\theta_1)}.\) The following table summarizes the above derivations.

**Per-unit royalty rate**

<table>
<thead>
<tr>
<th>Partial coverage: (\theta_1 &lt; 5\theta_2/6)</th>
<th>Full coverage: (\theta_1 \geq 5\theta_2/6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\pi_{1,t} = \frac{\theta_2^3}{27d(\theta_2-\theta_1)})</td>
<td>(\pi_{1,t} = \frac{(2\theta_1-\theta_2)^3}{2d})</td>
</tr>
<tr>
<td>(\pi_{2,t} = \frac{7\theta_2^3}{324d(\theta_2-\theta_1)})</td>
<td>(\pi_{2,t} = \frac{(\theta_2-\theta_1)(4\theta_1-\theta_2)}{3d})</td>
</tr>
</tbody>
</table>

**Ad valorem royalty rate**

<table>
<thead>
<tr>
<th>Partial coverage: (\theta_1 &lt; \theta_2/2)</th>
<th>Full coverage: (\theta_1 \geq \theta_2/2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\pi_{1,v} = \frac{\theta_2^3}{24d(\theta_2-\theta_1)})</td>
<td>(\pi_{1,v} = \frac{4\theta_2^2-2\theta_1\theta_2+\theta_2^2}{12d})</td>
</tr>
<tr>
<td>(\pi_{2,v} = \frac{\theta_2^3}{48d(\theta_2-\theta_1)})</td>
<td>(\pi_{2,v} = \frac{4\theta_2^2-2\theta_1\theta_2+\theta_2^2}{24d})</td>
</tr>
</tbody>
</table>

Using this table we can compare the profits of the firms in three regions:

**Region 1:** Partial coverage under both royalty rates (\(\theta_1 < \theta_2/2\)).

In this region, \(\pi_{1,t} = \frac{\theta_2^3}{27d(\theta_2-\theta_1)} < \frac{\theta_2^3}{24d(\theta_2-\theta_1)} = \pi_{1,v},\) so the licensor prefers an ad valorem rate. \(\pi_{2,t} = \frac{7\theta_2^3}{324d(\theta_2-\theta_1)} > \frac{\theta_2^3}{48d(\theta_2-\theta_1)} = \pi_{2,v},\) so the licensee prefers a per-unit rate.

**Region 2:** Partial coverage under per-unit rate and full coverage under ad valorem rate (\(\theta_2/2 \leq \theta_1 < 5\theta_2/6\)).

In this region, for the licensor, \(\pi_{1,v} - \pi_{1,t} = \frac{4\theta_2^2-2\theta_1\theta_2+\theta_2^2}{12d} - \frac{\theta_2^3}{27d(\theta_2-\theta_1)} = \frac{36\theta_2^2\theta_2-18\theta_1\theta_2+9\theta_2^2-36\theta_2^2+18\theta_2^2-9\theta_2^2-4\theta_2^3}{108d(\theta_2-\theta_1)}\)
rate when positive in valorem rate when prefers an ad valorem rate for equation 36
the difference 7+
3
12
. Since \( \theta_2/2 \leq \frac{3+\sqrt{3}}{6} \theta_2 < 5\theta_2/6 \), we conclude that the licensor prefers an ad valorem rate for \( \theta_2/2 \leq \theta_1 < \frac{3+\sqrt{3}}{6} \theta_2 \) and prefers a per-unit rate for \( \frac{3+\sqrt{3}}{6} \theta_2 \leq \theta_1 < 5\theta_2/6 \).

For the licensee, \( \pi_{2,t} - \pi_{2,v} = \frac{7\theta_3^2}{324d(\theta_2-\theta_1)} - \frac{4\theta_1^2 - 2\theta_1 \theta_2 + \theta_3^2}{24d} = \frac{14\theta_1^2 - 108\theta_2^2 \theta_3 + 54\theta_1 \theta_2^2 - 27\theta_3^2 + 108\theta_2^2 - 54\theta_1 \theta_2^2 + 27\theta_1 \theta_2^2}{648d(\theta_2-\theta_1)} \).

For \( \theta_1 > \theta_2 \), \( 3\theta_1 - \theta_2 \) is always positive, so \( \pi_{2,t} - \pi_{2,v} > 0 \), and the licensee always prefers a per-unit rate in this region.

Region 3: Full coverage under both royalty rates (\( \theta_1 \geq 5\theta_2/6 \)).

In this region, for the licensor \( \pi_{1,t} - \pi_{1,v} = \frac{4\theta_1^2 - 4\theta_1 \theta_2 + \theta_3^2}{2d} - \frac{4\theta_1^2 - 2\theta_1 \theta_2 + \theta_3^2}{12d} = \frac{20\theta_1^2 - 22\theta_1 \theta_2 + 5\theta_3^2}{12d} \).

For the values of \( \theta_1 \geq 5\theta_2/6 \), this function is increasing (the derivative of the numerator is \( 40\theta_1 - 22\theta_2 \)), and at \( \theta_1 = 5\theta_2/6 \), its value is \( \theta_3^2(500/36 - 110/6 + 5) > 0 \). Hence, in this region, the difference \( \pi_{1,t} - \pi_{1,v} \) is always positive and the licensor prefers a per-unit rate. For the licensee, \( \pi_{2,v} - \pi_{2,t} = \frac{4\theta_1^2 - 24\theta_1 \theta_2 + 9\theta_3^2}{24d} - \frac{4\theta_1^2 - 5\theta_1 \theta_2 - \theta_3^2}{3d} = \frac{36\theta_1^2 - 429\theta_1 \theta_2 + 9\theta_3^2}{24d} \). The roots of the quadratic equation \( 36\theta_1^2 - 429\theta_1 \theta_2 + 9\theta_3^2 = 0 \) are \( \theta_1 = \frac{42 \pm \sqrt{1764 - 1296}}{72} \theta_2 = \frac{42 \pm \sqrt{168}}{72} \theta_2 = \frac{42 \pm 6\sqrt{13}}{72} \theta_2 = \frac{7+\sqrt{13}}{12} \theta_2 \). Since \( \frac{7+\sqrt{13}}{12} > \frac{5}{6} \), the difference \( \pi_{2,v} - \pi_{2,t} \) is negative in \( 5\theta_2/6; \frac{7+\sqrt{13}}{12} \theta_2 \) and positive in \( \frac{7+\sqrt{13}}{12} \theta_2; \theta_2 \). Thus, the licensee prefers a per-unit rate for \( 5\theta_2/6 \leq \theta_1 < \frac{7+\sqrt{13}}{12} \theta_2 \) and prefers an ad valorem rate for \( \theta_1 \geq \frac{7+\sqrt{13}}{12} \theta_2 \).

In summary, the licensor prefers an ad valorem rate when \( \theta_1 < \frac{3+\sqrt{3}}{6} \theta_2 \) and a per-unit rate when \( \theta_1 \geq \frac{3+\sqrt{3}}{6} \theta_2 \). The licensor prefers a per-unit rate when \( \theta_1 < \frac{7+\sqrt{13}}{12} \theta_2 \) and an ad valorem rate when \( \theta_1 \geq \frac{7+\sqrt{13}}{12} \theta_2 \).
1) Computation of the optimal strategies under an ad valorem rate with a cap for a continuum of consumer types.

When the licensee faces an ad valorem rate \( \bar{v} \) with a cap \( \bar{p} \), its price and quantity schedules will have different expressions depending on whether a consumer’s taste preference is below or above \( \hat{\theta} \), where \( \hat{\theta} \) is the lowest level for which \( p(\theta) \geq \bar{p} \). If \( \theta \geq \hat{\theta} \), then

\[
MR(\theta) = \theta - \left(1 - \frac{F(\theta)}{F(\theta)}\right) = 2\theta - \theta_2. 
\]

After equating it to \( MC = c + dq \) and simplifying we obtain

\[
q(\theta) = \frac{2\theta - \theta_2 - c}{d}. \tag{10}
\]

If \( \theta < \hat{\theta} \), then an incremental increase in the revenue from a higher level of quality is \((1 - v)f(\theta)\) while a corresponding drop in the revenue from having to charge smaller prices to the higher types is \((1 - \bar{v})(F(\hat{\theta}) - F(\theta)) + (1 - \bar{v}F(\hat{\theta}))\). Then,

\[
MR(\theta) = (1 - \bar{v})\theta - ((1 - \bar{v})(\hat{\theta} - \theta) + \theta_2 - \hat{\theta}) = (1 - \bar{v})(2\theta - \hat{\theta}) - \theta_2 + \hat{\theta}. 
\]

After equating it to \( MC \) and simplifying, we obtain

\[
q(\theta) = \frac{(1 - \bar{v})(2\theta - \hat{\theta}) - (\theta_2 - \hat{\theta} + c)}{d}. \tag{11}
\]

The price schedule is determined by

\[
p(\theta) = \theta q(\theta) - z(\theta), \quad z(\theta) = \int_{\hat{\theta}}^{\theta} q(s)ds. 
\]

If \( \theta \geq \hat{\theta} \),

\[
z(\theta) = \int_{\hat{\theta}}^{\theta} \frac{(1 - \bar{v})(2s - \hat{\theta}) - (\theta_2 - \hat{\theta} + c)}{d} ds + \int_{\hat{\theta}}^{\theta} \frac{2s - \theta_2 - c}{d} ds. 
\]

The first integral is equal to

\[
\frac{1}{d} \left( (1 - \bar{v}) \left( \hat{\theta}^2 - \theta^2 - \hat{\theta}^2 + \theta\hat{\theta} \right) - \theta_2\hat{\theta} + \hat{\theta}^2 - c\hat{\theta} + \theta_2\hat{\theta} - \theta\hat{\theta} + c\hat{\theta} \right). 
\]

The second integral is

\[
\frac{1}{d} \left( \theta^2 - \hat{\theta}^2 - \theta_2\hat{\theta} - c\theta + c\hat{\theta} \right). 
\]

Adding these integrals and simplifying, we obtain

\[
z(\theta) = \frac{1}{d} \left( (1 - v) \left( \hat{\theta}^2 - \theta^2 \right) + \theta^2 - \theta_2 - c\theta + \theta_2\hat{\theta} - \theta\hat{\theta} + c\hat{\theta} \right). 
\]

Substitute this expression and (10)
into \( p(\theta) = \theta q(\theta) - z(\theta) \) and simplify to get

\[
p(\theta) = \frac{\theta^2 + (1-\nu)\left(\theta^2 - \hat{\theta}\right) - \theta_2 \theta + \hat{\theta} \theta - c\theta}{d}.
\]  

(12)

If \( \theta < \hat{\theta} \), \( z(\theta) = \int_{\hat{\theta}}^{\theta} \frac{(1-\nu)(z_d - z_{\hat{\theta} + c})}{d} ds \). This integral is equal to

\[
\frac{1}{d} \left( (1-\nu) \left( \theta^2 - \hat{\theta}^2 - \hat{\theta} \hat{\theta} + \hat{\theta} \theta \right) - \theta_2 \theta + \hat{\theta} \theta - c\theta + \theta_2 \theta - \hat{\theta} \theta + c\theta \right).
\]

Substitute this expression and (11) into \( p(\theta) = \theta q(\theta) - z(\theta) \) and simplify to get

\[
p(\theta) = \frac{1}{d} \left( \theta^2 + \theta^2 - \hat{\theta} \hat{\theta} \right) - \theta_2 \theta + \hat{\theta} \theta - c\theta.
\]

(13)

The licensee’s profit is

\[
\int_{\hat{\theta}}^{\theta} \left[ (1-\nu)p(\theta) - cq(\theta) - d\frac{q(\theta)^2}{2} \right] f(\theta) d\theta + \int_{\hat{\theta}}^{\theta_2} \left[ p(\theta) - cq(\theta) - d\frac{q(\theta)^2}{2} - \nu \dot{p} \right] f(\theta) d\theta.
\]

Substitute (11) and (13) into the integrand of the first integral and simplify to obtain

\[
(1-\nu)p(\theta) - cq(\theta) - d\frac{q(\theta)^2}{2} = \frac{1}{d} \left\{ -(1-\nu)^2 \theta^2 - 2\nu(1-\nu)\hat{\theta} \theta + 2(1-\nu)\theta_2 \theta + (1-\nu)^2 \theta^2 + \nu(1-\nu)\hat{\theta} \theta - \right. \\
\left. (1-\nu)\theta_2 \theta + c + \nu \theta_2 \theta - \nu \theta^2 c - \frac{\theta^2}{2} + \frac{c^2}{2} \right\}.
\]

Integrating the above expression and simplifying gives us

\[
\int_{\hat{\theta}}^{\theta} \left[ (1-\nu)p(\theta) - cq(\theta) - d\frac{q(\theta)^2}{2} \right] f(\theta) d\theta 
\]

\[
= \frac{1}{d(\theta_2 - \theta_1)} \left\{ \frac{\nu^2 - \theta_2^2 - \nu^3}{6} - \frac{2(1-\nu)^2}{3} \theta^3 + \theta_2^2 \hat{\theta}^2 + (1-\nu)\theta_2^2 c + (1-\nu)^2 \theta_2^2 \hat{\theta} - \nu(1-\nu)\hat{\theta}^2 - \\
\theta_2 \theta \hat{\theta} - (1-\nu)\theta_2 \theta c - \frac{\theta_2 \hat{\theta}^2}{2} + \frac{\theta_2 c}{2} - \frac{\theta \hat{\theta}^2}{2} \right\}.
\]

Substitute (10) and (12) into the integrand of the second integral in the profit formula above and simplify to obtain

\[
p(\theta) - cq(\theta) - d\frac{q(\theta)^2}{2} - \nu \cdot \dot{p} = \frac{1}{d} \left[ -\theta^2 + 2\theta \dot{\theta} + \theta \hat{\theta} - \theta(\theta_2 + c) - \frac{\theta_2}{2} + \frac{c^2}{2} - d \cdot \nu \cdot \dot{p} \right].
\]
Integrating the above expression and simplifying gives us

\[
\int_{\theta_1}^{\theta_2} \left[ p(\theta) - cq(\theta) - d^2(\theta^2 - \bar{\theta}^2) - \bar{\varphi}p \right] d\theta = \frac{1}{d(\theta_2 - \theta_1)} \left[ \frac{\theta_2^3}{3} + \frac{\bar{\theta}^3}{3} - \theta_2\bar{\theta}^2 + (1 - \bar{\varphi})(\theta_2\bar{\theta}^2 - \bar{\theta}\theta^2) + (1 + \bar{\varphi})\theta_2\bar{\theta}\theta - \bar{\varphi}\theta_2\theta^2 - \theta_2(\theta_2 + c) + \frac{\theta_2^2\bar{\theta}c + \frac{(\theta_2 - \bar{\theta})c^2}{2} - (\theta_2 - \bar{\theta})d \cdot \bar{\varphi} \cdot \bar{p}}{2} \right].
\]  

(15)

Finally, adding (14) and (15) gives us the formula for the licensee’s profit:

\[
\pi_2 = \frac{1}{d(\theta_2 - \theta_1)} \left[ \frac{\theta_2^3}{6} + \frac{\bar{\theta}^3}{6} - \frac{2(1 - \varphi)^3}{3} - \varphi(1 - \bar{\varphi})\theta_2\theta^2 + (1 - \varphi)\theta_2\theta^2 - \bar{\varphi}(1 - \bar{\varphi})\theta^2\bar{\theta} - \frac{\theta_2^2\bar{\varphi}\theta + \theta\bar{\varphi}\bar{\theta}c - \frac{\theta_2^2\bar{\varphi}}{2} - \theta_2\theta c + \frac{(\theta_2 - \bar{\theta})c^2}{2} - (\theta_2 - \bar{\theta})d \cdot \bar{\varphi} \cdot \bar{p}}{2} \right].
\]  

(16)

The licensee will choose the lowest \( \theta \) that allows for a non-negative quantity. Use (11) to set up \( q(\theta) = \frac{(1 - \bar{\varphi})(2\theta - \bar{\varphi}) - (\theta_2 - \bar{\theta} + c)}{d(\theta_2 - \theta_1)} = 0 \), from where \( \theta = \frac{\theta_2 - \bar{\theta} + c}{2(1 - \bar{\varphi})} \). If \( \theta < \theta_1 \), the licensee will set \( \theta_1 \) as the lower bound for its strategy. Finally, the licensee will choose \( \bar{\theta} \) to maximize profit (16). This has to be done numerically.

The licensor’s profit is

\[
\pi_2 = \int_{\theta_1}^{\theta_2} \bar{\varphi} p(\theta) f(\theta) d\theta + \int_{\theta_1}^{\theta_2} \bar{\varphi} p f(\theta) d\theta.
\]

Substitute \( p(\theta) \) from (13) to obtain

\[
\pi_2 = \frac{\bar{\varphi}}{d(\theta_2 - \theta_1)} \left[ (1 - \bar{\varphi}) \left( \theta^2 + \theta^2 - \theta_2^2 - \bar{\varphi}\theta_2 + \theta_2\theta - \bar{\varphi}\theta \right) d\theta + \frac{\theta_2^2\theta_2}{2(1 - \bar{\varphi})} \bar{\varphi} \right].
\]

This profit is

\[
\pi_2 = \frac{\bar{\varphi}}{d(\theta_2 - \theta_1)} \left[ (1 - \bar{\varphi}) \left( \theta^2 + \theta^2 - \theta_2^2 - \bar{\varphi}\theta_2 + \theta_2\theta - \bar{\varphi}\theta \right) \right] + \frac{\theta_2^2\theta_2}{2(1 - \bar{\varphi})} \bar{\varphi}.
\]  

(17)

Taking into account the licensee’s response of \( \bar{\theta} \) and \( \theta_2 \), the licensor chooses \( \bar{\varphi} \) and \( \bar{p} \) to
maximize (17). This problem is also solved numerically.
Online Appendix B for Manuscript "Licensing Mechanisms for Product Lines"

Figure 1B: Market Coverage Under Different Types of Royalty Rates (Two Consumer Types)

Note. The dashed line separates the regions with different market coverage when there is no licensing.
Figure 2B: Licensee’s Preference for the Type of the Royalty Rate (Two Consumer Types)
Figure 3B: Licensor’s Strategy under an Ad Valorem Rate with a Cap (Two Consumer Types)

Note. The ad valorem rate increases with \( \theta_2 \).

price, quality

\( \bar{p}_2 \)
\( \bar{q}_2 \)
\( \bar{q}_1 = \bar{p}_2 \)
\( \theta_2 \)
Figure 4B: Licensor’s Preference for the Type of the Royalty Rate (Two Consumer Types)
Figure 5B: Market Coverage Under Different Types of Royalty Rates (Continuum of Consumer Types)

Note. The dashed line separates the regions with different market coverage when there is no licensing.
Figure 6B: Firms’ Preferences Between Per-Unit and Ad Valorem Rates (Continuum of Consumer Types)

Note. The dashed line in panel b) separates the regions with different preferences of the licensor from panel a).
Figure 7B: Licensor’s Preference for the Type of the Royalty Rate (Continuum of Consumer Types)