LECTURE 29: THE ESSENCE OF ANALYSIS

Video: [The Essence of Analysis](whole lecture)

There’s a saying in German that says: “Everything has an end, except for a sausage, which has two.” And with this, I would like to welcome you to the last lecture of this class and the last lecture of my UC Irvine career! In this segment called *The Essence of Analysis*, I will go over the main concepts of the course and explain how they’re tied together.

1. **Real Numbers**

Video: [Real Numbers](whole lecture)

Real Analysis is the study of real numbers. You might ask: “Why *Real* Analysis?” What is it about the real numbers that makes them so different from the rational numbers? In particular, why is there no course called *Rational Analysis*? For this, let’s discuss some number systems.

First, we define $\mathbb{N} = \{1, 2, \ldots \}$, which do inductively: Start with 1, then define 2 as the successor of 1, then 3 as the successor of 2 etc.

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*Date:* Friday, June 5, 2020.

1. *Alles hat ein Ende, nur die Wurst, die hat zwei*
The main problem with $\mathbb{N}$ is that we cannot subtract things, so numbers like $1 - 2 = -1$ are not in $\mathbb{N}$. That’s why we need to extend our universe to the integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots \}$

But then, the problem with $\mathbb{Z}$ is that we can’t divide things, so things like $\frac{1}{2}$ are not in $\mathbb{Z}$. Hence we extend out universe once again to get the rational numbers $\mathbb{Q}$, which are just fractions of the form $\frac{a}{b}$, where $a$ and $b$ are integers (and $b \neq 0$)

What makes $\mathbb{Q}$ nice is that it’s a field, which is a system where you can add/subtract/multiply/divide by things.

Even more is true: $\mathbb{Q}$ is an ordered field, meaning that we can compare rational numbers, like $\frac{1}{2} \leq \frac{2}{3}$.

Not every number is rational. For instance, $\sqrt{2}$ is irrational. However, this doesn’t explain why $\mathbb{R}$ is better than $\mathbb{Q}$ because, by the same token, not every number is real, so we should study only complex numbers or quaternions.
The reason we study \( \mathbb{R} \) is because of the **least upper bound property**, which we’ll define below. A least upper bound is a generalization of a maximum, so let’s define that first.

**Definition:**

Let \( S \) be a nonempty subset of \( \mathbb{R} \). Then \( M = \max S \) means:

1. \( M \in S \)
2. For all \( s \in S \), \( s \leq M \)

The problem is that \( \max S \) doesn’t always exist! For example \( S = [0, 2) \) has no max because the max would be \( M = 2 \), which is not in \( S \)

**Question:** Is there a way of generalizing the max so that it always exists? Yes, and it’s the concept of a sup

In the picture above, the blue \( M \) on the far right is an upper bound of \( S \) (= for all \( s \in S \), \( s \leq M \)). But we can do better! The green \( M \)
to the second right is also a upper bound of \( S \), and smaller than the first one. And we can continue until we reach the *least* upper bound \( \sup(S) \) (which barely touches \( S \))

**Definition:**

\[
M = \sup(S) \text{ (} M \text{ is the *supremum* of } S \text{) if}
\]

1. \( M \) is an upper bound of \( S \)
2. For all \( M_1 < M \) there is \( s_1 \in S \) such that \( s_1 > M_1 \)

**Analogy:** If you (= \( M_1 \)) are not the best student (= \( M \)), then there is a student (= \( s_1 \)) who’s better than you.

This says that \( \sup(S) \) is the *least* upper bound of \( S \). Any other number \( M_1 \) smaller than the sup cannot be an upper bound.

**Question:** Does \( \sup(S) \) always exists? It depends!

In \( \mathbb{Q} \), \( \sup(S) \) does not always exist
Example:

Let \( S = \{ r \in \mathbb{Q} \mid r^2 < 2 \} = (-\sqrt{2}, \sqrt{2}) \) (in \( \mathbb{Q} \)).

In this case, \( \sup(S) = \sqrt{2} \), which is not in \( \mathbb{Q} \), so, strictly speaking, \( \sup(S) \) does not exist (in \( \mathbb{Q} \)). However, if you consider the same set in \( \mathbb{R} \), then \( \sup(S) \) exists, and is \( \sqrt{2} \).

IN FACT, the single, most important property that distinguishes \( \mathbb{R} \) from \( \mathbb{Q} \) (and the main reason that we study Real Analysis) is that \( \mathbb{R} \) satisfies the least-upper bound property, whereas \( \mathbb{Q} \) doesn’t:
Least Upper Bound Property

If \( S \neq \emptyset \) is bounded above, then \( \operatorname{sup}(S) \) exists

(If we include the case \( \operatorname{sup}(S) = \infty \), then we don’t even need to require that \( S \) be bounded above)

It seems like such a small detail, but it’s literally the Big Bang that triggers the Theory of Real Analysis; and in fact, without it, we wouldn’t even be here!

Some Applications:

Archimedean Property

If \( a > 0 \) and \( b > 0 \) are real numbers, then there is \( n \in \mathbb{N} \) such that \( na > b \)

\[ a \quad 2a \quad 3a \quad b \quad na \]

\[ a \quad a \quad a \quad a \]

\( \mathbb{Q} \) is dense in \( \mathbb{R} \)

For any \( a, b \in \mathbb{R} \) with \( a < b \) there is \( r \in \mathbb{Q} \) such that \( a < r < b \)
A couple of Remarks:

(1) Similarly to sup, we can define the greatest lower bound of $S$

\[
m = \inf(S) \text{ if for all } m_1 > m \text{ there is } s_1 \in S \text{ such that } s_1 < m_1
\]
However, inf and sup are related via

\[ \inf(S) = - \sup(-S) \]

In particular, if \( S \) is bounded below, \( \inf(S) \) exists.

(3) We can extend the real numbers to include \( \pm \infty \). In this case

**Definition:**

\( \sup(S) = \infty \) if for all \( M \) there is \( s_1 \in S \) such that \( s_1 > M \).
Finally, there is a way of explicitly constructing \( \mathbb{R} \) using Dedekind cuts, but this is beyond the scope of the exam. That said, there is a beautiful comment (from a YouTuber) that perfectly summarizes what \( \mathbb{R} \) is:

**Quote:** A real number is a way of cutting up (= Dedekind cuts) equivalence classes (= Rational Numbers) of equivalence classes (= Integers) of ways of building up the empty set (= Natural Numbers)

## 2. Sequences

**Video:** [Sequences]

Now that we’ve defined real numbers, we can talk about sequences, which are infinite lists of real numbers.

![Sequence Diagram]

The single most important thing about sequences is the definition of convergence:

**Definition:**

\[
\lim_{n \to \infty} s_n = s \text{ if and only if } \\
\text{For all } \epsilon > 0 \text{ there is some } N \text{ such that if } n > N, \text{ then } |s_n - s| < \epsilon
\]
(No matter how small the error, there is some threshold \( N \) such that, after that threshold, \( s_n \) is within \( \epsilon \) away from \( s \))

**Examples:**

1. \( s_n = \frac{1}{n^2} \to 0 \)
2. \( s_n = \left(1 + \frac{1}{n}\right)^n \to e \)
3. \( s_n = (-1)^n \) does not converge

From this, we can prove some nice limit laws, such as
Note: Similarly, we can define

Definition:
\[
\lim_{n \to \infty} s_n = \infty \text{ if and only if for all } M > 0 \text{ there is } N \text{ such that if } n > N, \text{ then } s_n > M
\]

In general, it is hard to show that a sequence converges, but luckily there are two neat convergence “tests:”
Cauchy:

\((s_n)\) is **Cauchy sequence** if for all \(\epsilon > 0\) there is \(N\) such that if \(m, n > N\), then

\[|s_m - s_n| < \epsilon\]

Monotone Sequence Theorem:

\((s_n)\) is increasing and bounded above, then \((s_n)\) converges.
Similarly if \((s_n)\) is decreasing and bounded below, then \((s_n)\) converges.

**Limsup:** The Monotone Sequence Theorem allows us to define the limsup, which is the analog of the sup, but for sequences. For this, let \((s_n)\) be a sequence and consider:

\[
\sup \{s_n \mid n > N\}
\]

(that is, the sup of \(s_n\) but after the threshold \(N\))

Notice that, the bigger \(N\), the smaller the sup is:
This is because, the bigger $N$, the less values of $s_n$ with $n > N$ there are, so the sup is becoming smaller (for example, if you have a class of 10 students, and 5 good students drop, then the highest score will be lower)

In particular the sequence $\sup \{ s_n \mid n > N \}$ is decreasing, and so, by the Monotone Sequence Theorem, it converges. It’s that limit that we call $\lim\sup$:

\[
\lim\sup s_n = \lim_{n \to \infty} \sup_{N \to \infty} \{ s_n \mid n > N \}
\]

And the amazing fact is that $\lim\sup$ always exists (or is $\pm\infty$)
Even though the lim sup is an abstract concept, we can always attain it using subsequences:

**Fact:**
There is a subsequence \((s_{n_k})\) of \((s_n)\) that converges to \(\limsup_{n \to \infty} s_n\)

(in other words, there is an express train with destination \(\limsup\))

Finally, the most important fact about subsequences is the Bolzano-Weierstraß Theorem:

**Bolzano-Weierstraß:**
Every bounded sequence \((s_n)\) has a convergent subsequence \((s_{n_k})\)
It says that if a sequence is bounded, it is trapped, which forces a subsequence to converge. That subsequence is like a string to a balloon, which prevents it from flying away.

3. Metric Spaces

Video: [Metric Spaces]

Now the question is, can we generalize all the results that we’ve learned so far to more general spaces? Yes, we can! For this, we need to define the concept of a metric space.

Notice: The only properties about absolute value that we’ve used are the following:

For all $x, y, z \in \mathbb{R}$,

1. $|x - y| \geq 0$ and $|x - y| = 0 \iff x = y$
2. $|x - y| = |y - x|$
3. **Triangle Inequality:** $|x - z| \leq |x - y| + |y - z|$
(Interpretation: The third leg of a triangle is less than or equal to the sum of the other two legs)

Definition:

$(S, d)$ is a **metric space** if:

1. $d(x, y) \geq 0$, $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Notice the similarity between $d(x, y)$ and $|x - y|$, so in fact $d$ (called a metric) is just a generalization of the absolute value.

Examples:

1. $\mathbb{R}$ (with the usual absolute value)
2. $\mathbb{R}^k$ (with the usual distance)
3. $\mathbb{Q}$
4. Any set with the discrete metric
5. Sequences
6. Continuous functions

Once we have a metric, we can define **convergence** of sequences in metric spaces, it’s amazing how it’s nearly identical our usual notion of convergence:
Definition:
If \((S, d)\) is a metric space and \((s_n)\) is a sequence in \(S\), then \(s_n \rightarrow s\) if for all \(\epsilon > 0\) there is \(N\) such that if \(n > N\), then \(d(s_n, s) < \epsilon\) 

And similar for Cauchy sequences:

Definition:
\((s_n)\) is Cauchy if, for all \(\epsilon > 0\), there is \(N\) such that if \(m, n > N\), then \(d(s_n, s_m) < \epsilon\)
Although Convergence ⇒ Cauchy, it is NOT always true that Cauchy ⇒ Convergence. But if it’s true, then we call that complete:

**Definition:**

\[(S,d) \text{ is complete} \text{ if every Cauchy sequence } (s_n) \text{ in } S \text{ converges}\]

So, for instance, \(\mathbb{R}\) and \(\mathbb{R}^k\) are complete. This is yet another reason why \(\mathbb{R}\) is so much better than \(\mathbb{Q}\).

**Open Sets:** A very useful set in \(\mathbb{R}\) is the open interval, or (more generally) the open ball:

**Definition:**

\[B(x,r) = \{y \in S \mid d(x,y) < r\}\]

And using balls, we can define the notion of an open set:
Definition:

$E$ is **open** if for all $x \in E$ there is $r > 0$ such that $B(x, r) \subseteq E$.

In other words, at any point in $E$, we can squeeze in a very small ball. So open sets have a little wiggle room, we can always move around them.

And similarly, there is the notion of a closed set, but for this we have to define **limit points**:

Definition:

$s \in \overline{E}$ if and only if there is a sequence $(s_n)$ in $E$ that converges to $s$.

In some sense $\overline{E}$ is the set of all the places you can reach using sequences.
Definition:

$E$ is closed if and only if $\overline{E} = E$, that is, whenever $(s_n)$ is a sequence in $E$ that converges to $s$, then $s \in E$.

So a closed set contains all its limit points; you cannot escape $E$ by taking limits.

**Compactness:** Another useful set is the closed ball $\overline{B(x,r)}$. What makes closed balls so nice is that they are closed and bounded, and compactness is the perfect generalization of closed and bounded sets:
Definition:
A set $E$ is **compact** if every open cover $\mathcal{U}$ of $E$ has a finite subcover $\mathcal{V}$.

**Fact 1:**
If $E$ is compact, then it is closed and bounded.

In $\mathbb{R}^k$, the converse is true as well:

**Fact 2: Heine-Borel Theorem**
In $\mathbb{R}^k$, $E$ is compact if and only if $E$ is closed and bounded.

(But in general metric spaces, this is not true, see Problem 15 in HW7)

**Cantor Set:** Finally, a wonderful example of a compact set in $\mathbb{R}$ is the Cantor set, which is obtained by starting with $[0,1]$ and successively...
removing the middle third of each interval:

4. Series

Video: [Series]

Given a sequence \((a_n)\), we would like to sum up all the values of \(a_n\). For this, we need to use partial sums:

**Definition:**

\[
\sum_{n=0}^{\infty} a_n = S \iff \lim_{n \to \infty} s_n = S , \text{ where } \\
s_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n
\]
Notice that indeed \( s_n = a_0 + \cdots + a_n \) is a sequence depending only on \( n \). In other words, the series \( \sum a_n \) converges if and only if the sequence \((s_n)\) \textit{converges}. This ties a new concept with an old concept that we already know.

**Definition:**

If the above limit exists, then we say \( \sum a_n \) \textbf{converges}. Else, if \( S = \pm \infty \) and/or the limit does not exist, then \( \sum a_n \) \textbf{diverges}.

**Example:**

What is \( \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots \)?

For this, look at

\[
s_n = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \left(\frac{1}{2}\right)} = \frac{1}{1 - \frac{1}{2}} = 2
\]

Therefore \( \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2 \)

In general, it is difficult to find the exact value of a series; that’s why, in the remainder, we will just focus on the easier problem of figuring out if a series converges. For this, there are again two neat tricks:

**Fact:**

Suppose \( a_n \geq 0 \) for all \( n \)

Then \( \sum a_n \) converges if and only if \((s_n)\) (as above) is \textbf{bounded}.
Note: This is what calculus textbooks mean when they say “A series converges if and only if it is bounded”

This is because, since \( a_n \geq 0 \), \((s_n)\) is a non-decreasing sequence, and therefore, \((\Leftarrow)\) \((s_n)\) is bounded, by the Monotone Sequence Theorem, \((s_n)\) converges, which implies that \(\sum a_n\) converges.

The second trick is the Cauchy criterion:

**Cauchy Criterion:**

\[ \sum_{n=1}^{\infty} a_n \text{ converges if and only if it satisfies the Cauchy criterion:} \]

For all \(\epsilon > 0\) there is \(N\) such that, if \(n \geq m > N\), then

\[ \left| \sum_{k=m}^{n} a_k \right| < \epsilon \]

In other words, no matter how small the error, any tail \(\sum_{k=m}^{n} a_k\) of the series (no matter how long) eventually becomes as small as we want.
Convergence Tests: Then there are all the convergence tests from Calculus:

1. **Divergence Test** (If \((a_n)\) doesn’t converge, then \(\sum a_n\) diverges)

2. **Comparison Test** (If \(0 \leq a_n \leq b_n\) and \(\sum b_n\) converges, then \(\sum a_n\) converges)

3. **Ratio Test** (If \(\limsup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \alpha\), then if \(\alpha < 1\), \(\sum a_n\) converges absolutely and if \(\alpha > 1\), then \(\sum a_n\) diverges)

4. **Root Test** (If \(\limsup_{n \to \infty} |a_n|^\frac{1}{n} = \alpha\), then if \(\alpha < 1\), \(\sum a_n\) converges absolutely and if \(\alpha > 1\), then \(\sum a_n\) diverges)

5. **Integral Test**

6. **Alternating Series Test** (For series of the form \(\sum (-1)^n a_n\), like \(1 - \frac{1}{2} + \frac{1}{3} - \ldots\). If \(a_n \geq 0\) is decreasing and \(\to 0\), then \(\sum a_n\) converges, very easy to apply)
(Notice how in the Ratio and Root tests we use lim sup. This is because for example we don’t know if the limit of $|a_n|^{\frac{1}{n}}$ exists)

An important consequence from the Integral Test is:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges } \iff p > 1$$

5. **Continuity**

**Video:** [Continuity](#)

Finally, we would like to define the notion of continuity, which just means that if the inputs of $x$ are close together, then the outputs $f(x)$ are close together.

There are two equivalent definitions of continuity:

**Definition 1:**

$f$ is **continuous at** $x_0$ if, whenever $x_n$ is a sequence that converges to $x_0$, then $f(x_n)$ converges to $f(x_0)$
Definition 2:

\( f \) is \textit{continuous} at \( x_0 \) if for all \( \epsilon > 0 \) there is \( \delta > 0 \) such that for all \( x \), if \( |x - x_0| < \delta \), then \( |f(x) - f(x_0)| < \epsilon \)

\( f \) is \textit{continuous} if for all \( x_0 \), \( f \) is continuous at \( x_0 \)

(No matter how small the error \( \epsilon \), there is some small threshold \( \delta \) such that, if \( |x - x_0| \) is within the threshold \( \delta \), then \( |f(x) - f(x_0)| \) is within...
the good region $\epsilon$)

Using those definitions, one can show that familiar functions such as $x^2, e^x, \sin(x)$ are continuous.

Moreover, one can show some continuity laws, such as, whenever $f$ and $g$ are continuous, then $f + g, fg, \frac{f}{g}, g \circ f, f^{-1}$ are continuous

**Properties of Continuity:** Continuous functions enjoy some nice properties:

**Fact 1:**
If $f : [a, b] \to \mathbb{R}$ is continuous, then $f$ is bounded

![Continuous vs Not Continuous]

**Fact 2:** Extreme Value Theorem:
Suppose $f : [a, b] \to \mathbb{R}$ is continuous, then $f$ has a maximum and a minimum on $[a, b]$
Fact 3: Intermediate Value Theorem:

If $f : [a, b] \to \mathbb{R}$ is continuous and if $c$ is any number between $f(a)$ and $f(b)$, then there is some $x \in [a, b]$ such that $f(x) = c$.

Uniform Continuity: This is a better version of continuity, and basically says that $\delta$ doesn’t depend on $x_0$, it doesn’t depend on where
your position.

**Definition:**

*f* is uniformly continuous on a set *S* if for all *ε* > 0 there is *δ* > 0 such that, for all *x, y ∈ S*, if |*x* − *y| < *δ*, then |*f*(*x*) − *f*(*y*)| < *ε*

(Here *δ* does not depend on *x* or *y*; there is universal *δ* that works for every *x* and *y*)
Useful Properties:

(1) If \( f : [a, b] \to \mathbb{R} \) is continuous, then \( f \) is uniformly continuous on \([a, b]\)

(2) If \( f \) is uniformly continuous on \( S \) and \((s_n)\) is Cauchy, then \( f(s_n) \) is Cauchy (not true if \( f \) is just continuous)

(3) \( f : (a, b) \to \mathbb{R} \) is uniformly continuous on \((a, b)\) if and only if \( f \) has a continuous extension \( \tilde{f} : [a, b] \to \mathbb{R} \)

The last property says that if \( f \) is uniformly continuous on \((a, b)\), then we can extend \( f \) to a continuous function at the endpoints.

Examples:

(1) \( f(x) = x \sin \left( \frac{1}{x} \right) \) is uniformly continuous on \((0, 1]\) because we can extend it to a continuous function \( \tilde{f} \) on \([0, 1]\) (by defining \( \tilde{f}(0) = 0 \))

(2) \( f(x) = \sin \left( \frac{1}{x} \right) \) is not uniformly continuous on \((0, 1]\) because we can’t extend \( f \) to a continuous function \( \tilde{f} \) on \([0, 1]\)
6. **EPILOGUE**

Video: [Good Bye](#)

**CONGRATULATIONS**, you have now officially reached a limit point of your Analysis adventure! I just wanted to thank you and all of UC Irvine for the wonderful 3 years I have had here, I will always cherish them deeply in my heart. With that said, thank you so much for flying Peyam Airlines, I hope you had a pleasant stay on board, and I wish you a safe onward journey! **ZOT ZOT ZOT!!!**

The End