Reading: Sections 18 and 19. In section 19, ignore Discussion 19.3 with the integrals.

- **Section 18:** 9, 10, 12, AP1 (Optional: AP2, AP3, AP4, AP5, AP6, AP7, AP8, AP9, AP10)

- **Section 19:** 2, 4, 5, 6, 7, 8, 9, 11

Additional Problem 1: Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(f(f(x))) = x$ for all $x$, then $f(x) = x$.

Note: The Optional Additional Problems are on the next page. They discuss (among other things) the concept of a homeomorphism. You may have heard of the expression *A coffee cup is like a donut.* That has to do with homeomorphisms.

CONGRATULATIONS!!! You are now officially done with the homework! Words cannot express how proud I am of you, this was no easy task at all, but you’ve reigned supreme 😊

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Date: Due: Thursday, June 4, 2020.
Definition:
Let $E$ be any subset of $\mathbb{R}$ (or of any metric space)

1. $E$ is **disconnected** if there are disjoint, nonempty, and open subsets $A$ and $B$ of $E$ such that $A \cup B = E$
2. $E$ is **connected** if it is not disconnected

For example, $\mathbb{R}$ is connected but $(0, 1) \cup (2, 3)$ is disconnected

Optional Additional Problem 2: Use the hints to give a 2-line proof of the Intermediate Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous and $c$ is between $f(a)$ and $f(b)$, then there is $x \in [a, b]$ with $f(x) = c$. Isn’t connectedness awesome?

Optional Additional Problem 3: Suppose $E$ is connected and $f : E \to \mathbb{R}$ is continuous, prove that $f(E)$ is connected.

Optional Additional Problem 4: Prove that $\mathbb{R}$ is connected (see hints) (More generally, it follows that any interval $I$ is connected)

Definition:
Let $E$ be any subset of $\mathbb{R}$ (or of any metric space)

1. A path in $E$ is a continuous function $\gamma : [0, 1] \to E$
2. $E$ is **path-connected** if for any pair of points $a$ and $b$ in $E$, there is a path $\gamma$ with $\gamma(0) = a$ and $\gamma(1) = b$

Optional Additional Problem 5:

(a) Show that if $E$ is path-connected, then it is connected
(b) Show $\mathbb{R}$ is path-connected and deduce that it is connected.

Optional Additional Problem 6: The topologist’s sine curve is defined as

$$E = F \cup G =: \left\{ \left( x, \sin \left( \frac{1}{x} \right) \right) \mid x \in (0, 1) \right\} \cup \{0\} \times [-1, 1]$$

Show that $E$ is connected but not path-connected.

Optional Additional Problem 7: Use the result of AP7(a) in HW9 and AP2 in HW7 to give a quick proof of the Extreme Value Theorem: If $K$ is a compact subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ is continuous, then $f$ attains a maximum and a minimum
Definition:

Let $A$ and $B$ be two subsets of $\mathbb{R}$ (or any two metric spaces) and $f : A \rightarrow B$ is a function, then:

(a) $f$ is a **homeomorphism** if $f$ is continuous, one-to-one, onto, and $f^{-1}$ is continuous

(b) $A$ and $B$ are **homeomorphic** if there is a homomorphism between $A$ and $B$

(c) A **topological property** is a property that is preserved under homeomorphisms

Optional Additional Problem 8:

(a) Show that there is a homeomorphism between $(0, 1)$ and $\mathbb{R}$. So surprisingly $(0, 1)$ and $\mathbb{R}$ are homeomorphic

(b) Deduce that boundedness is not a topological property.

Optional Additional Problem 9:

(a) Show that if $f : I \rightarrow f(I)$ is continuous and one-to-one, then $f$ is a homeomorphism

(b) Show that if $K$ is covering compact and $f : K \rightarrow f(K)$ is continuous and one-to-one, then $f$ is a homeomorphism

(c) Let $S^1$ be the unit circle in $\mathbb{R}^2$. Consider the map $f : [0, 2\pi) \rightarrow S^1$ by $f(t) = (\cos(t), \sin(t))$. You may assume that $f$ is continuous, one-to-one, and onto. Show that $f^{-1}$ is not continuous and hence not a homeomorphism.

Optional Additional Problem 10:
(a) Show that homeomorphisms map compact sets onto compact sets. Hence compactness is a topological property. Deduce that $[0, 1]$ and $\mathbb{R}$ are not homeomorphic.

(b) Show that homeomorphisms map connected sets onto connected sets. So connectedness is a topological property. Deduce that $[0, 2\pi]$ and the unit circle $S^1$ in $\mathbb{R}^2$ are not homeomorphic.

(c) Show openness and closedness are topological properties. Deduce that $(0, 1)$ and $[0, 1]$ (considered as subsets of $\mathbb{R}$) are not homeomorphic.
Hints:

18.9 Let \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \). WLOG, we may assume \( a_n > 0 \). Since \( f \) goes to \( \infty \) as \( x \) goes to \( \infty \), there is \( b > 0 \) large enough such that \( f(b) > 0 \), and since \( f \) goes to \( -\infty \) as \( x \) goes to \( -\infty \), there is \( a < 0 \) such that \( f(a) < 0 \). There is no need to prove those statements since we haven’t defined limits yet. The book’s hint is a bit confusing in my opinion.

18.10 Argue in cases, if \( f(1) > f(0) \) or \( f(1) = f(0) \) or \( f(1) < f(0) \)

18.12 Suppose \( a < b \), then if \( 0 < a < b \), then since \( f \) is continuous on \([a, b]\) we can just apply the IVT, and similarly if \( a < b < 0 \), so the interesting case is \( a \leq 0 \leq b \). WLOG, assume \( b > 0 \). It might be helpful to solve for \( \sin \left( \frac{1}{x} \right) = 1 \) and \( \sin \left( \frac{1}{x} \right) = -1 \)

19.4(a) By “assume not”, the book means that there is a sequence \((x_n)\) in \( S \) such that \( |f(x_n)| \to \infty \). Remember that Cauchy sequences are bounded.

19.6(a) Use that \( f(x) = \sqrt{x} \) is continuous on \([0, \infty)\), hence uniformly continuous on \([0, 1]\) (which is compact)

19.9 You don’t need to reprove the result in \((a)\). As far as \((b)\) is concerned, if \( S \) is bounded, notice that \( \overline{S} \) (the closure of \( S \)) is closed and bounded in \( \mathbb{R} \) and therefore compact. Then use Theorem 19.2. For \((c)\), don’t be discouraged; it’s not as tricky as the book makes it seem. You don’t need to reprove why it suffices to show that \( f \) is uniformly continuous on \((-\infty, 1]\) and \([1, \infty)\) since you’ve already done this in 19.7(a), but please show that \( f \) is uniformly continuous on \([1, \infty)\) (the other part is similar).
19.11 Again, no need to reprove that why it suffices to show that \( f \) is uniformly continuous on \(( -\infty, 1]\) and \([1, \infty)\) since you’ve already done this in 19.7(a), but please show that \( f \) is uniformly continuous on \([1, \infty)\) (the other part is similar).

**AP 1:** First show that \( f \) must be one-to-one. For this suppose \( f(x) = f(y) \) and apply \( f \) twice to this equation. Therefore, \( f \) must be increasing or decreasing. But if \( f \) is decreasing, suppose \( x < y \) and apply \( f \) three times to get a contradiction. Hence \( f \) is increasing. Now if \( f(x) \neq x \) for some \( x \), then either \( f(x) > x \) or \( f(x) < x \). In both cases, apply \( f \) twice to get a contradiction.

**Note:** You can find solutions to this problem in the following video: Press fff to pay respects

**AP 2:** Suppose \( f : [a, b] \rightarrow \mathbb{R} \) is continuous but there is \( c \) such that \( f(x) \neq c \) for all \( c \). Then consider the sets:

\[
A = \{ x \in [a, b] \mid f(x) < c \} = f^{-1}((-\infty, c))
\]
\[
B = \{ x \in [a, b] \mid f(x) > c \} = f^{-1}((c, \infty))
\]

For this, use the awesome definition of continuity in HW9, as well as AP6 in HW 9. You may assume that \([a, b]\) is connected

**Note:** Even though the sets are not open in \( \mathbb{R} \), they are open in \([a, b]\)

**AP 3:** Suppose not, and consider \( f^{-1}(A \cup B) \). For this, use the awesome definition of continuity in HW9, as well as AP6 in HW 9

**AP 4:**
Suppose \( \mathbb{R} \) is not connected. Then we can write \( \mathbb{R} = A \cup B \) with \( A, B \) nonempty, open and disjoint.

**STEP 1:** Since \( A \) and \( B \) are nonempty, fix \( a \in A \) and \( b \in B \). WLOG \( a < b \) and let

\[
S = \{ x \in [a, b] \mid [a, x] \subseteq A \}
\]

Show \( S \) has a least upper bound \( M =: \sup(S) \)

**STEP 2:**

**Claim:** \( M \notin B \)

Suppose \( M \in B \). Then since \( B \) is open, there is \( r > 0 \) such that \((M - r, M + r) \subseteq B\).

Use the definition of \( S \) to conclude that \( M \notin B \) and conclude that \( M \in A \)

**STEP 3:** Moreover \( M \in S \), because if \( M \notin S \), then \([a, M]\) \( \not\subseteq A \), meaning there is \( x \in [a, M] \) with \( x \notin A \). But since \( M \in A \), we have \( x < M = \sup(S) \) and therefore there is \( y \in S \) with \( y > x \).

Find a contradiction

**STEP 4:** Show \( M < b \). For this, assume \( b \leq M \) and show that \( b \in A \)

**STEP 5:**

**Claim:** \( M \notin A \)
Suppose $M \in A$, then, since $A$ is open, there is $r' > 0$ such that $(M - r', M + r') \subseteq A$. Let $M' = \min \{M + r', b\}$

Then $M' > M$, and so $M' \notin S$ because $M = \sup(S)$.

Therefore, by definition of $S$, $[a, M'] \not\subseteq A$, so there is some $x \in [a, M']$ with $x \notin A$. But since $[a, M] \subseteq A$ (because $M \in S$), we must have $x \in (M, M']$.

Show $x \in A$ and find a contradiction, and conclude that $\mathbb{R}$ must be connected

**AP 5:** Suppose $E$ is path-connected but not connected. Pick $a \in A$ and $b \in B$, and consider $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$

**AP 6:** Hints to be posted this week-end 😊

**AP 7:**

**Proof that $E$ is connected:** First notice that $\overline{F} = E$ and show the following, more general result (which is true in any metric space)

**Claim:** If $F$ is connected subset of $\mathbb{R}^2$, then $\overline{F}$ is connected

For this, notice that the result is true of $F = \emptyset$, so assume $F \neq \emptyset$, suppose $F$ is connected but $\overline{F}$ is not connected. Then there are open nonempty disjoint subsets $A$ and $B$ of $\overline{F}$ such that $A \cup B = \overline{F}$.

Consider $A' = A \cap F$ and $B' = B \cap F$.

Show WLOG, assume $A' = F$ and $B' = \emptyset$
Notice that, since $A^c = B$ is open (the complement here is in $\overline{F}$) we get $A$ is closed in $\overline{F}$

Now by definition of $A$ closed in $\overline{F}$, there is some closed subset $C$ of $\mathbb{R}^2$ with $A = C \cap \overline{F}$ (we haven’t covered relative closedness in the course, so take that as a given)

**Use the definition of $\overline{F}$ to conclude that $A = \overline{F}$ and find a contradiction.**

**Proof that $E$ is not path-connected:** Suppose not, then in particular is there is $\gamma : [0, 1] \to E$ with $\gamma(0) \in F$ and $\gamma(1) \in G$.

**Why is it ok to assume that $\gamma(1) = (0, 1)$?**

Let $\epsilon = \frac{1}{2}$, then by continuity of $\gamma$ at 1, there is $\delta > 0$ such that

If $|t - 1| \leq \delta \Rightarrow 1 - \delta \leq t \leq 1$, then

$$|\gamma(t) - \gamma(1)| < \frac{1}{2} \Rightarrow |\gamma(t) - (0, 1)| < \frac{1}{2}$$

(Note: Here the absolute value for $\gamma$ is just the usual distance in $\mathbb{R}^2$. Also the $\leq \delta$ isn’t really a problem)

Let $\gamma(1 - \delta) =: (x_0, y_0)$ and remember that $\gamma(1) = (0, 1)$

**Use the Intermediate Value Theorem to show that $\gamma_1([1 - \delta, 1])$ contains the interval $[0, x_0]$, where $\gamma = (\gamma_1, \gamma_2)$**

Hence for all $x_1 \in (0, x_0]$ there is some $t$ with $\gamma_1(t) = x_1$ and therefore, by definition, there is $t \in [1 - \gamma, 1]$ such that
\[ \gamma(t) = (\gamma_1(t), \gamma_2(t)) = (x_1, \sin \left( \frac{1}{x_1} \right)) \]

Find \( x_1 \) such that \( 0 < x_1 < x_0 \), but \( \sin \left( \frac{1}{x_1} \right) = \sin \left( -\frac{\pi}{2} \right) = -1 \)

Conclude that the \( t \) you found is a distance of \( \frac{1}{2} \) away from \((0,1)\) and find a contradiction.

**AP 8:** Your solution might involve \( \tan^{-1} \)

**AP 9(a):** It’s one of the theorems in lecture

**AP 9(b):** All you need to check is that \( f^{-1} \) is continuous. For this, show that it’s enough to show that whenever \( C \) is closed, then \( f(C) \) is closed. It might also be useful to show that \((f^{-1})^{-1}(C) = f(C)\) (that is the preimage of \( C \) when you apply \( f^{-1} \) to is \( f(C) \)). Now if \( C \) is a closed subset of a compact set, it is compact, and then use the result of AP7 in HW9.

**AP 9(c):** Find a sequence \( (x_n) \) of points on the circle that converges to \( f(0) = (1,0) \) but such that \( f^{-1}(x_n) \) converges to \( \frac{\pi}{2} \)

**AP 10(b):** Even though \([0,2\pi]\) and \( S^1 \) are connected, what happens if you remove 1 from \([0,2\pi]\), is it still connected? What if you remove a point from \( S^1 \)?