**AP 1**

First, let’s show $f$ is one-to-one. Suppose $f(x) = f(y)$, then $f(f(x)) = f(f(y))$, so $f(f(f(x))) = f(f(f(y)))$, hence $x = y \checkmark$

Now since $f$ is continuous and one-to-one on $\mathbb{R}$, $f$ must be either increasing or decreasing.

But if $f$ were decreasing, then if $a < b$, then $f(a) > f(b)$ so $f(f(a)) < f(f(b))$ and hence $f(f(f(a))) > f(f(f(b)))$, hence $a > b \Rightarrow\Leftarrow$

Hence $f$ is increasing.

Now suppose $f(x) \neq x$, then either $f(x) < x$ or $f(x) > x$.

But if $f(x) < x$, then $f(f(x)) < f(x)$ so $f(f(f(x))) < f(f(x))$ and therefore $x < f(f(x))$, and so we get:

$$x < f(f(x)) < f(x) < x \Rightarrow\Leftarrow$$

And we get a similar contradiction if $f(x) > x \Rightarrow\Leftarrow$.

Therefore we must have $f(x) = x$ for all $x$ \hfill \square

---

*Date: Due: Thursday, June 4, 2020.*
AP 2
Suppose not, then there is \( c \) such that \( f(x) \neq c \) for all \( x \in [a, b] \). This means that for all \( x \), either \( f(x) > c \) or \( f(x) < c \), and therefore \([a, b] = A \cup B\) where

\[
A = \{x \in [a, b] \mid f(x) < c\} = f^{-1}((-\infty, c)) \\
B = \{x \in [a, b] \mid f(x) > c\} = f^{-1}((c, \infty))
\]

Now \( A \cup B = \emptyset \) and \( A \) and \( B \) are nonempty since either \( f(a) \) or \( f(b) \) are in \( A \) or \( B \)

Moreover, \( A \) and \( B \) are open since \( f \) is continuous and \((-\infty, c)\) and \((c, \infty)\) are open.
And therefore \([a, b] = A \cup B\) with \( A \) and \( B \) nonempty, open, and disjoint, which contradicts the fact that \([a, b] \) is connected. \( \Rightarrow \Leftarrow \) □

AP 3
Suppose \( E \) is connected by \( f(E) \) is not connected. Then there are \( A \) and \( B \) nonempty, open, and disjoint with \( f(E) = A \cup B \).

But now consider \( A' = f^{-1}(A) \) and \( B' = f^{-1}(B) \). Then, since \( A \) and \( B \) are open and \( f \) is continuous, we get \( A' \) and \( B' \) are open. Moreover:

\[
A' \cap B' = f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset \\
A' \cup B' = f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(f(E)) = E
\]
(The latter follows because for all \( x \in E \), \( f(x) \in f(E) \) and therefore \( x \in f^{-1}(f(E)) \))
Finally, since $A$ is nonempty there is $a \in A \subseteq f(E)$ and therefore there is $a' \in E$ with $f(a') \in A$ and so $a' \in f^{-1}(A) = A'$ and so $A'$ is nonempty, and similarly $B'$ is nonempty.

Therefore $A'$ and $B'$ are disjoint, nonempty, and open subsets of $E$ with $A' \cup B' = E$, but this implies that $E$ is disconnected $\Rightarrow \Leftarrow$

AP 4

Suppose $\mathbb{R}$ is not connected. Then we can write $\mathbb{R} = A \cup B$ with $A, B$ nonempty, open and disjoint.

STEP 1: Since $A$ and $B$ are nonempty, fix $a \in A$ and $b \in B$. WLOG $a < b$ ($a \neq b$ since $A$ and $B$ are disjoint) and consider

$$S = \{x \in [a, b] \mid [a, x] \subseteq A\}$$

Then $S$ is nonempty since $a \in S$ and moreover $S$ is bounded above by $b$, hence $S$ has a least upper bound $M = \sup(S)$

STEP 2:

Claim: $M \notin B$

Suppose $M \in B$. Then since $B$ is open, there is $r > 0$ such that $(M - r, M + r) \subseteq B$.

Since $M - r < M = \sup(S)$, there is $x \in S$ such that $x > M - r$. Since $x \in S$, we get $[a, x] \subseteq A$, and so $x \in A$. But, on the other hand $x \in (M - r, M] \subseteq (M - r, M + r) \subseteq B$, and therefore $x \in A \cap B = \emptyset \Rightarrow \Leftarrow$. Hence, since $M \notin B$ and $A \cup B = \mathbb{R}$, we must have $M \in A$
**STEP 3:** Moreover $M \in S$, because if $M \not\in S$, then $[a, M] \not\subseteq A$, meaning there is $x \in [a, M]$ with $x \notin A$. But since $M \in A$, we have $x < M = \sup(S)$ and therefore there is $y \in S$ with $y > x$. But by definition of $S$, we have $[a, y] \subseteq A$ and so, since $x < y$ we get $[a, x] \subseteq [a, y] \subseteq A$, which is a contradiction since $x \notin A$.

**STEP 4:** Now $M < b$, because if $b \leq M$, then we get a contradiction because, since $M \in S$, we have $[a, M] \subseteq A$ and so $b \in [a, M] \subseteq A$ so $b \in A \Rightarrow \Leftarrow$

**STEP 5:**

**Claim:** $M \notin A$

Suppose $M \in A$, then, since $A$ is open, there is $r' > 0$ such that $(M - r', M + r') \subseteq A$. Let $M' = \min \{M + r', b\}$

Then $M' > M$, and so $M' \notin S$ because $M = \sup(S)$.

Therefore, by definition of $S$, $[a, M'] \not\subseteq A$, so there is some $x \in [a, M']$ with $x \notin A$. But since $[a, M] \subseteq A$ (because $M \in S$), we must have $x \in (M, M']$. Moreover, $x \neq M + r'$ (because $M + r' \in A$ but $x \notin A$), and therefore $x \in (M, M + r') \subseteq A$, so $x \in A \Rightarrow \Leftarrow$.

Hence $M \notin A$ either, and therefore $M$ is neither in $A$ or in $B$, which contradicts $\mathbb{R} = A \cup B \Rightarrow \Leftarrow$. □

**AP 5**

For (a), suppose $E$ is path-connected but not connected. Since $E$ is not connected, there are $A$ and $B$, nonempty, open, and disjoint such
that $A \cup B = E$.

Since $A$ and $B$ are nonempty, there is $a \in A$ and $b \in B$.

Since $\gamma$ is path-connected, there is a path $\gamma : [0, 1] \to E$ with $\gamma(0) = a$ and $\gamma(1) = b$.

Now consider $A' = \gamma^{-1}(A)$ and $B' = \gamma^{-1}(B)$. Then since $A$ and $B$ are open and $\gamma$ is continuous, we get $A'$ and $B'$ are open.

Moreover $0 \in A'$ since $\gamma(0) = a \in A$ and therefore $A'$ is nonempty, and similarly $B'$ is nonempty, and finally

$$A' \cap B' = \gamma^{-1}(A' \cap B') = \gamma^{-1}(A') \cap \gamma^{-1}(B') = A \cap B = \emptyset$$

$$A' \cup B' = \gamma^{-1}(A' \cup B') = \gamma^{-1}(A') \cup \gamma^{-1}(B') = A \cup B = [0, 1]$$

But therefore $A'$ and $B'$ are disjoint, open, nonempty subsets of $[0, 1]$ whose union in $[0, 1]$, which contradicts that $[0, 1]$ is connected $\Rightarrow \Leftarrow$.

Hence $E$ must be connected.

For (b), let $a, b \in \mathbb{R}$ and consider the path $\gamma(t) = (1 - t)a + tb$, which is continuous and has values in $\mathbb{R}$ and $\gamma(0) = a$ and $\gamma(1) = b$ $\checkmark$

**AP 6**

**Note:** The solutions here are taken from this handout

**Proof that $E$ is connected:**
Claim: If $F$ is connected subset of $\mathbb{R}^2$, then $\overline{F}$ is connected

Proof: The result is true of $F = \emptyset$, so assume $F \neq \emptyset$.

Suppose $F$ is connected but $\overline{F}$ is not connected. Then there are open nonempty disjoint subsets $A$ and $B$ of $\overline{F}$ such that $A \cup B = \overline{F}$.

Consider $A' = A \cap F$ and $B' = B \cap F$. Then $A'$ and $B'$ are open in $F$, disjoint, and their union is $F$. But since $F$ is connected, we must have $A' = F$ and $B' = \emptyset$ or $A' = \emptyset$ and $B' = F$.

WLOG, assume $A' = F$ and $B' = \emptyset$

Notice that, since $A^c = B$ is open (the complement here is in $\overline{F}$) we get $A$ is closed in $\overline{F}$, hence there is some closed subset $C$ of $\mathbb{R}^2$ with $A = C \cap \overline{F}$

But then $F = A' \subseteq A \subseteq C$, and since $\overline{F}$ is the smallest closed subset containing $F$, we get $\overline{F} \subseteq C$, and hence

$$A = C \cap \overline{F} = \overline{F}$$

And since $A$ is disjoint from $B$, we must get that $B = \emptyset \Rightarrow \Leftarrow$. $\square$

To show that our topologist sine curve is connected, let

$$F = \left\{ \left(x, \sin \left(\frac{1}{x}\right) \right) \mid x \in (0, 1] \right\}$$

Then $F$ is connected since it is path-connected (for $a$ and $b$ in $(0, 1]$, just consider the path $\gamma(t) = \left((1 - t)a + tb, \sin \left(\frac{1}{(1-t)a+tb}\right) \right)$), and moreover $\overline{F} = E$ (since $\sin \left(\frac{1}{x}\right)$ has the intermediate value property on $\mathbb{R}$), and
therefore by the Claim, $E$ is connected

**Proof that $E$ is not path-connected:** Suppose not, then in particular is there is $\gamma : [0, 1] \to E$ with $\gamma(0) \in F$ and $\gamma(1) \in G$.

Because $G$ is just a straight line (which is path-connected), we may assume $\gamma(1) = (0, 1)$.

Let $\epsilon = \frac{1}{2}$, then by continuity of $\gamma$ at 1, there is $\delta > 0$ such that

- If $|t - 1| \leq \delta \Rightarrow 1 - \delta \leq t \leq 1$, then
  
  $$|\gamma(t) - \gamma(1)| < \frac{1}{2} \Rightarrow |\gamma(t) - (0, 1)| < \frac{1}{2}$$

(Note: Here the absolute value for $\gamma$ is just the usual distance in $\mathbb{R}^2$. Also the $\leq \delta$ isn’t really a problem)

Let $\gamma(1 - \delta) =: (x_0, y_0)$ and remember that $\gamma(1) = (0, 1)$

Since $\gamma = (\gamma_1, \gamma_2)$ is continuous, the first component $\gamma_1$ is continuous, and therefore, by the Intermediate Value Theorem, $\gamma_1$ attains all the values between $\gamma_1(1 - \delta) = x_0$ and $\gamma_1(1) = 0$, and hence $\gamma_1([1 - \delta, 1])$ contains the interval $[0, x_0]$.

Hence for all $x_1 \in (0, x_0]$ there is some $t$ with $\gamma_1(t) = x_1$ and therefore, by definition, there is $t \in [1 - \gamma, 1]$ such that

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)) = \left(x_1, \sin \left(\frac{1}{x_1}\right)\right)$$

But now let $x_1 = \frac{1}{2\pi n - \frac{\pi}{2}}$, then for $n$ large enough we have $0 < x_1 < x_0$, but $\sin \left(\frac{1}{x_1}\right) = \sin \left(-\frac{\pi}{2}\right) = -1$
Hence the point \( \left( \frac{1}{2\pi n - \frac{\pi}{2}}, -1 \right) \) has the form \( f(t) \) for some \( t \in [1 - \delta, 1] \) and hence \( t \) is a distance of \( \frac{1}{2} \) away from \((0, 1)\), which contradicts the fact that the distance between \( \left( \frac{1}{2\pi n - \frac{\pi}{2}}, -1 \right) \) and \((0, 1)\) is at least 2.

\[ \implies \square \]

\[ \text{AP 7} \]

Since \( K \) is compact, and \( f \) is continuous, \( f(K) \) is compact by AP7(a) in HW9. Since \( f(K) \) is compact, it is closed and bounded, and therefore it has a least upper bound \( M = \sup(f(K)) \).

Let \((y_n)\) be a sequence in \( f(K) \) converging to \( M \). By definition of \( f(K) \), \( y_n = f(x_n) \) for some \( x_n \in K \).

But since \( K \) is (covering) compact, \( K \) is sequentially compact, and therefore \( (x_n) \) has a convergent subsequence \( (x_{n_k}) \) that converges to some \( x_0 \in K \).

But since \( f \) is continuous, we get \( f(x_{n_k}) \to f(x_0) \).

But then since \( y_n \) converges to \( M \), the subsequence \( y_{n_k} = f(x_{n_k}) \) converges to \( M \), so by uniqueness of limits, \( f(x_0) = M \), so \( f \) has a maximum \( M \) at \( x_0 \in K \), and similarly \( f \) has a minimum \( m \) at some other point. \[ \square \]

\[ \text{AP 8} \]

For (a), consider \( f : (0, 1) \to \mathbb{R} \) defined by

\[ f(x) = \tan^{-1}\left( \pi x - \frac{\pi}{2} \right) \]
Then, one can check that \( g(x) = \pi x - \frac{\pi}{2} \) is continuous, one-to-one, and onto, and its inverse is continuous and therefore a homeomorphism.

Also since \( \tan : \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \to \mathbb{R} \) is continuous and one-to-one and onto \( \mathbb{R} \) (you can show this using the fact that \( \tan(x) \to \pm\infty \) near \( \pm\frac{\pi}{2} \) and an analog of the Intermediate Value Theorem), its inverse \( \tan^{-1} \) is continuous, and therefore a homeomorphism.

Hence \( f(x) \) is a homeomorphism, being a composition of two homeomorphisms, and therefore \((0,1)\) and \( \mathbb{R} \) are homeomorphic.

For (b), since \((0,1)\) is bounded but \( \mathbb{R} \) is unbounded, boundedness is not a topological property.

**AP 9**

(a) By assumption, \( f \) is continuous, one-to-one, and onto its image \( f(I) \). Moreover, we have shown in class that \( f^{-1} \) is continuous, hence \( f \) is a homeomorphism.

(b) Since \( f \) is continuous, one-to-one, and onto its image, it suffices to show that \( f^{-1} \) is continuous.

**Claim:** \( f \) is continuous if and only if for each closed set \( C \), \( f^{-1}(C) \) is closed

This follows because if \( f \) is continuous and \( C \) is closed, then \( C^c \) is open, and therefore \( f^{-1}(C^c) \) is open, hence \( (f^{-1}(C))^c \) is open, so \( f^{-1}(C) \) is closed ✓
Conversely, if $f^{-1}(C)$ is closed whenever $C$ is closed, then if $U$ is any open set, then $U^c$ is closed, so by assumption $f^{-1}(U^c)$ is closed, and therefore $(f^{-1}(U))^c$ is closed, and so $f^{-1}(U)$ is open, so $f$ is continuous $\checkmark$

Now suppose $C$ is an arbitrary closed subset of $K$, then since $K$ is compact, $C$ is a closed subset of a compact set, and hence compact. Therefore, since $C$ is compact and $f$ is continuous, $f(C)$ is compact, and hence closed.

Therefore, whenever $C$ is closed, $f(C)$ is closed, and by the claim below, it follows that $(f^{-1})^{-1}(C) = f(C)$ is closed, and so $f^{-1}$ is continuous since $f$ was arbitrary

**Claim:** $(f^{-1})^{-1}(C) = f(C)$

**Proof:**

$$
x \in (f^{-1})^{-1}(C) \iff f^{-1}(x) \in C
\iff f \left( f^{-1}(x) \right) \in f(C)
\iff x \in f(C) \checkmark \quad \Box
$$

(c) Let

$$(x_n) = \left( \cos \left( 2\pi - \frac{1}{n} \right), \sin \left( 2\pi - \frac{1}{n} \right) \right)$$

Then $(x_n)$ converges to $(1, 0)$, but $f^{-1}(x_n) = 2\pi - \frac{1}{n}$ converges to $2\pi \neq f^{-1}((1, 0)) = 0$. 
Hence $f^{-1}$ is not continuous.

AP 10
(a) This just follows because if $K$ is compact and $f$ is continuous, then $f(K)$ is compact. Therefore, since $[0,1]$ is compact but $\mathbb{R}$ is not compact, the two spaces are not homeomorphic.

(b) This just follows because if $E$ is connected and $f$ is continuous, then $f(E)$ is connected.

It is not hard to show that if $f : E \to F$ is a homeomorphism and $x_0 \in E$, then $f : E \setminus \{x_0\} \to F \setminus \{f(x_0)\}$ is also a homeomorphism.

Now if $[0,2\pi]$ and $S^1$ were homeomorphic with homeomorphism $f$, then $[0,2\pi] \setminus \{1\}$ and $S^1 \setminus \{f(1)\}$ would also be a homeomorphism. But this can’t be because $[0,2\pi] \setminus \{1\} = [0,1) \cup (1,2\pi]$ is disconnected, whereas $S^1$ minus a point is still connected! $\Rightarrow \Leftarrow$

(c) Suppose $A$ is open and $f$ is a homeomorphism, then $f(A) = (f^{-1})^{-1}(A)$ is open since $f^{-1}$ is continuous and $A$ is open. Similarly, if $B$ is closed, then $f(B) = (f^{-1})^{-1}(B)$ is closed since $f^{-1}$ is continuous and $B$ is closed.

Now Since $(0,1)$ is open in $\mathbb{R}$ and $[0,1]$ is not open in $\mathbb{R}$, those two cannot be homeomorphic.