Let $M = \sup(S)$. Then, for every $n$, $M - \frac{1}{n} < M = \sup(S)$, and therefore, by definition of sup, there is $s_n \in S$ such that $s_n > M - \frac{1}{n}$. But since $M$ is an upper bound for $S$, we also have $s_n \leq M$, and therefore $M - \frac{1}{n} < s_n \leq M$

Since $M - \frac{1}{n} \rightarrow M$ and $M \rightarrow M$, by the squeeze theorem, we have $s_n \rightarrow M = \sup(S)$
2.

Let $\epsilon > 0$ be given.

Since $s_n \to s$, there is $N_1$ such that if $n > N_1$, then $|s_n - s| < \frac{\epsilon}{3}$

Since $t_n \to t$, there is $N_2$ such that if $n > N_2$, then $|t_n - t| < \frac{\epsilon}{3}$

Since $u_n \to u$, there is $N_3$ such that if $n > N_3$, then $|u_n - u| < \frac{\epsilon}{3}$

Let $N = \max\{N_1, N_2, N_3\}$, then if $n > N$, we get:

$$|s_n + t_n + u_n - (s + t + u)| = |s_n - s + t_n - t + u_n - u|$$

$$\leq |s_n - s| + |t_n - t| + |u_n - u|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

$$= \epsilon \sqrt{3}$$

Hence $s_n + t_n + u_n \to s + t + u$ \qed
3.

(a) Let $N$ be given, then if $n > N$, we have

$$s_n \geq \inf \{ s_n \mid n > N \}$$

And

$$t_n \geq \inf \{ t_n \mid n > N \}$$

And therefore

$$s_n + t_n \geq \inf \{ s_n \mid n > N \} + \inf \{ t_n \mid n > N \}$$

Since $n > N$ was arbitrary, taking the inf over all $n > N$ on the left hand side, we get

$$\inf \{ s_n + t_n \mid n > N \} \geq \inf \{ s_n \mid n > N \} + \inf \{ t_n \mid n > N \}$$

And therefore

$$\liminf_{n \to \infty} (s_n + t_n) = \liminf_{N \to \infty} \inf \{ s_n + t_n \mid n > N \}$$

$$\geq \liminf_{N \to \infty} \inf \{ s_n \mid n > N \} + \inf \{ t_n \mid n > N \}$$

$$= \liminf_{N \to \infty} \inf \{ s_n \mid n > N \} + \liminf_{N \to \infty} \inf \{ t_n \mid n > N \}$$

$$= (\liminf_{n \to \infty} s_n) + (\liminf_{n \to \infty} t_n)$$
(b) Let \( s_n = (-1)^n \) and \( t_n = -s_n = (-1)^{n+1} \)

Then

\[
\left( \liminf_{n \to \infty} s_n \right) + \left( \liminf_{n \to \infty} t_n \right) = -1 + (-1) = -2
\]

But

\[
\liminf_{n \to \infty} s_n + t_n = \liminf_{n \to \infty} 0 = 0 \neq -2
\]
4. Let \((x^{(n)}) = (x^{(n)}_1, x^{(n)}_2)\) be a Cauchy sequence in \(\mathbb{R}^2\)

**Claim:** \(x^{(n)}_1\) and \(x^{(n)}_2\) are Cauchy in \(\mathbb{R}\)

**Proof:** Let \(\epsilon > 0\) be given, then there is \(N\) such that if \(m, n > N\), then \(d_\infty(x^{(m)}, x^{(n)}) < \epsilon\), that is

\[
\max \left\{ \left| x^{(m)}_1 - x^{(n)}_1 \right|, \left| x^{(m)}_2 - x^{(n)}_2 \right| \right\} < \epsilon
\]

With that same \(N\), if \(m, n > N\), then

\[
\left| x^{(m)}_1 - x^{(n)}_1 \right| \leq \max \left\{ \left| x^{(m)}_1 - x^{(n)}_1 \right|, \left| x^{(m)}_2 - x^{(n)}_2 \right| \right\} < \epsilon
\]

And similarly \(\left| x^{(m)}_2, x^{(n)}_2 \right| < \epsilon\). \(\square\)

Since \(x^{(n)}_1\) is Cauchy and \(\mathbb{R}\) is complete, \(x^{(n)}_1 \rightarrow x_1\) for some \(x_1 \in \mathbb{R}\), and similarly \(x^{(n)}_2 \rightarrow x_2\) for some \(x_2 \in \mathbb{R}\). Let \(x =: (x_1, x_2)\)

**Claim:** \(x^{(n)} \rightarrow x\)

Let \(\epsilon > 0\) be given. Then, since \(x^{(n)}_1 \rightarrow x_1\), there is \(N_1\) such that if \(n > N_1\), then \(\left| x^{(n)}_1 - x_1 \right| < \epsilon\), and similarly there is \(N_2\) such that if \(n > N_2\), then \(\left| x^{(n)}_2 - x_2 \right| < \epsilon\).

Let \(N = \max \left\{ N_1, N_2 \right\}\), then if \(n > N\), we have

\[
d_\infty(x^{(n)}, x) = \max \left\{ \left| x^{(n)}_1 - x_1 \right|, \left| x^{(n)}_2 - x_2 \right| \right\} < \max \{\epsilon, \epsilon\} = \epsilon
\]

And therefore \(x^{(n)} \rightarrow x\). \(\square\)
5. (a) Let \( f(x) = \frac{1}{x^2} \), then \( f \) is \( \geq 0 \) and decreasing and

\[
\int_{1}^{\infty} f(x) \, dx = \int_{1}^{\infty} \frac{1}{x^2} \, dx = \left[ -\frac{1}{x} \right]_{1}^{\infty} = 0 + 1 = 1 < \infty
\]

Therefore by the Integral Test, \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges.

(b) Let \( \epsilon > 0 \) be given. Then since \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges, it satisfies the Cauchy criterion, and hence there is \( N \) such that if \( n \geq m > N \), then

\[
\left| \sum_{k=m}^{n} \frac{1}{k^2} \right| < \epsilon
\]

But then, with the same \( N \), if \( m, n > N \), WLOG, \( n > m \), and therefore

\[
|s_m - s_n| = |s_m - s_{m+1} + s_{m+1} - s_{m+2} + \cdots + s_{n-1} - s_n| \\
\leq |s_m - s_{m+1}| + |s_{m+1} - s_{m+2}| + \cdots + |s_{n-1} - s_n| \\
\leq \frac{1}{m^2} + \frac{1}{(m+1)^2} + \cdots + \frac{1}{(n-1)^2} \\
= \sum_{k=m}^{n-1} \frac{1}{k^2} \\
< \sum_{k=m}^{n} \frac{1}{k^2} \\
< \epsilon \sqrt{n}
\]

Hence \( (s_n) \) is Cauchy, and since \( \mathbb{R} \) is complete, \( (s_n) \) converges. \( \square \)
6. **BOLZANO-WEIERSTRAẞ TIME!!!**

Since \((s_n)\) is a sequence in \([a, b]\), \((s_n)\) is bounded, and therefore, by the Bolzano-Weierstraß Theorem, \((s_n)\) has a convergent subsequence \((s_{n_k})\) that converges to some \(x \in [a, b]\).

Since \(s_{n_k} \to x\) and \(f\) is continuous, \(f(s_{n_k}) \to f(x)\).

But, on the other hand, since \(0 \leq f(s_n) \leq \frac{1}{n}\), we have \(0 \leq f(s_{n_k}) \leq \frac{1}{n_k}\), and \(\frac{1}{n_k} \to 0\), by the squeeze theorem, we have \(f(s_{n_k}) \to 0\).

Combining \(f(s_{n_k}) \to f(x)\) and \(f(s_{n_k}) \to 0\), we get \(f(x) = 0\). □
\begin{enumerate}
\item
\textbf{STEP 1: Scratchwork:}

\[ |f(x) - f(y)| \leq C |x - y|^\alpha < \epsilon \Rightarrow |x - y|^\alpha < \frac{\epsilon}{C} \Rightarrow |x - y| < \left( \frac{\epsilon}{C} \right)^{\frac{1}{\alpha}} \]

\textbf{STEP 2: Actual Proof:}

Let \( \epsilon > 0 \) be given, let \( \delta = \left( \frac{\epsilon}{C} \right)^{\frac{1}{\alpha}} \), then if \( x, y \in \mathbb{R} \) with \( |x - y| < \delta \), then

\[ |f(x) - f(y)| \leq C |x - y|^\alpha < C \left[ \left( \frac{\epsilon}{C} \right)^{\frac{1}{\alpha}} \right]^\alpha = C \left( \frac{\epsilon}{C} \right) = \epsilon \]

Therefore \( f \) is uniformly continuous on \( \mathbb{R} \) \qedhere
\end{enumerate}
8.

By assumption, for all $x \in S$, there are $M = M(x) > 0$ and $r = r(x) > 0$ such that $|f(y)| \leq M(x)$ for all $y \in B(x, r(x))$.

Now consider the following family of sets

$$\mathcal{U} = \{B(x, r(x)) \mid x \in S\}$$

Each $B(x, r(x))$ is open (by definition) and each $x \in S$ is in $B(x, r(x))$, hence $\mathcal{U}$ is an open cover of $S$.

But since $S$ is compact, $\mathcal{U}$ has a finite sub-cover

$$\mathcal{V} = \{B(x_1, r(x_1)), \ldots, B(x_N, r(x_N))\}$$

Let $M = \max \{M(x_1), \ldots, M(x_N)\}$ (which is independent of $x$)

Then for all $x \in S$, since $\mathcal{V}$ covers $S$, we have $x \in B(x_n, r(x_n))$ for some $n$, and therefore, by definition

$$|f(x)| \leq M(x_n) \leq M \Rightarrow |f(x)| \leq M \sqrt{\phantom{x}}$$