1. (10 points)

Suppose $S$ is a nonempty subset of $\mathbb{R}$ with $\sup(S) = \infty$ (this means that for all $M$, there is $s \in S$ with $s > M$)

Use an inductive construction to find an increasing sequence $(s_n)$ in $S$ such that $s_n > n$ for all $n \in \mathbb{N}$
2. (10 points)

Show directly (without using lim sup or the ratio or root tests) that if \((a_n)\) is a sequence of non-negative real numbers with

\[
\liminf_{n \to \infty} |a_n|^\frac{1}{n} = \alpha > 1
\]

Then \(\sum a_n = \infty\)

**Note:** Start by letting \(\epsilon > 0\) be such that \(\alpha - \epsilon > 1\). Here you want the inf to be *bigger* than something.
3. \((10 = 4 + 4 + 2 \text{ points})\)

Let \(E\) be the following subset of \(\mathbb{R}\) (with its usual metric):

\[
E = \left\{ 2 - \left( \frac{1}{n} \right) \mid n \in \mathbb{N} \right\}
\]

Find the following, with justification:

(a) The interior \(E^\circ\) of \(E\)
(b) The closure \(\overline{E}\) of \(E\) (the book uses \(E^-\))
(c) The boundary \(\partial E\) of \(E\)
4. (10 points)

Let \((S, d)\) be a compact metric space (not necessarily in \(\mathbb{R}\) or \(\mathbb{R}^k\)) and let \(F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots\) be a non-increasing sequence of nonempty closed sets \(F_n\).

Show that the intersection \(\bigcap_{n=1}^{\infty} F_n\) is nonempty.

**Hint:** Suppose \(\bigcap_{n=1}^{\infty} F_n = \emptyset\) and consider the open cover

\[ \mathcal{U} = \{F_1^c, F_2^c, \ldots\} \]

What set does \(\mathcal{U}\) cover? (Here \(F_n^c\) is the complement of \(F_n\) in \(S\). Notice that \(F_1^c \subseteq F_2^c \subseteq \ldots\))