LECTURE 8: THE WAVE EQUATION

Readings:

• Section 2.1: Transport Equation
• The Wave Equation (pages 65-66)
• Section 2.4.1a: D’Alembert’s Formula
• Section 4 of the Lecture Notes: Some consequences
• Section 2.4.1b: Spherical Means

Welcome to the final equation of this course: The Wave Equation

Wave Equation:

\[ u_{tt} = \Delta u \]

Compare this with the heat equation \( u_t = \Delta u \). Even though they look similar, they actually have different properties!

1. **The Transport Equation**

Reading: Section 2.1: The Transport Equation

Video: [Transport Equation](#)
Let’s first solve a related PDE that will be useful in our solution of the wave equation.

**Transport Equation:**

\[
\begin{aligned}
\begin{cases}
    u_t + b \cdot Du = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
    u(x, 0) = g(x)
\end{cases}
\end{aligned}
\]

**Example:** In 2 dimensions with \( b = (3, -2) \), this becomes

\[ u_t + 3u_{x_1} - 2u_{x_2} = 0 \]

It turns out this is fairly easy to solve: First of all, the equation \( u_t + b \cdot Du = 0 \) is suggesting that \( u \) is constant on lines directed by \( \langle b, 1 \rangle \), which are parametrized by \( (x + sb, t + s) \).

Therefore, if you let \( z(s) = u(x + sb, t + s) \), then

\[ z'(s) = u_{x_1}b_1 + \cdots + u_{x_n}b_n + u_t = u_t + b \cdot Du = 0 \]

Therefore \( z(s) \) is constant on lines, and hence in particular we get
\[ z(0) = z(-t) \]
\[ \Rightarrow u(x + 0b, t + 0) = u(x - tb, t - t) \]
\[ \Rightarrow u(x, t) = u(x - tb, 0) \]
\[ \Rightarrow u(x, t) = g(x - tb) \]

**Transport Equation:**

The solution of the following PDE is

\[
\begin{align*}
    u_t + b \cdot Du &= 0 \\
    u(x, 0) &= g(x)
\end{align*}
\]

\[ u(x, t) = g(x - tb) \]

Similarly, we get:

**Inhomogeneous version:**

The solution of the following PDE is

\[
\begin{align*}
    u_t + b \cdot Du &= f(x, t) \\
    u(x, 0) &= g(x)
\end{align*}
\]

\[ u(x, t) = g(x - tb) + \int_0^t f(x + (s - t)b, s) ds \]

The proof is the same, except here we don’t get \( z' = 0 \), but \( z' = f \) (and so \( z = \int f \))

2. **The Wave Equation**
**Reading:** Section 2.4: The Wave Equation (pages 65-66)

### Wave Equation:

\[ u_{tt} = \Delta u \]

### Derivation:

Similar to Laplace’s equation or the heat equation, except here you start with the identity \( F = ma \) (Force = mass times acceleration)

### Applications:

The applications of the wave equation depend on the dimension:

1. (1 dimension) Models a vibrating string: \( u(x, t) \) is the height of the string at position \( x \) and time \( t \)

   ![Wave Equation Diagram](image)

   Also used to model sound waves and light waves

2. (2 dimensions) Models *water waves*. For example, the wave equation models the water ripples caused by throwing a rock at a pond.
Also used to model a vibrating drum.

(3) (3 dimensions) Models vibrating solids, think like an elastic ball that vibrates

3. **D’ALEMBERT’S FORMULA** ($n = 1$)

**Reading:** Section 2.4.1a: D’Alembert’s Formula

**Video:** [D’Alembert’s Formula](#)

Although Laplace’s Equation and the Heat Equation were similar, the Wave equation is *very* different. It not only has different properties, but the derivation is also different.
What makes this even more interesting is that the derivation is different depending on the dimension: We will first do the 1–dimensional case, then (next time) the 3–dimensional case, and the 2–dimensional case.

Goal: \((n = 1)\)

Solve:

\[
\begin{align*}
  u_{tt} &= u_{xx} \\
  u(x, 0) &= g(x) \\
  u_t(x, 0) &= h(x)
\end{align*}
\]

(Vibrating string with initial position \(g(x)\) and initial velocity \(h(x)\))

**STEP 1: Clever Observation:** We can write \(u_{tt} - u_{xx} = 0\) as

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) u = 0
\]

In particular, if you let \(v = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) u = u_t - u_x\), then the above becomes

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) v = 0 \Rightarrow v_t + v_x = 0 \quad \text{TRANSPORT EQUATION!}
\]

Moreover:

\[
v(x, 0) = u_t(x, 0) + u_x(x, 0) = h(x) - (g(x))_x = h(x) - g'(x)
\]

**STEP 2:** Therefore we need to solve

\[
\begin{align*}
  v_t + v_x &= 0 \\
  v(x, 0) &= h(x) - g'(x)
\end{align*}
\]
(Transport equation with \( b = 1 \)), which gives:

\[
v(x,t) = h(x - tb) - g'(x - tb) = h(x - t) - g'(x - t)
\]

**STEP 3:** Solve for \( u \) using \( v = u_t - u_x \), that is:

\[
\begin{align*}
  u_t - u_x &= v = \underbrace{h(x - t) - g'(x - t)}_{f(x,t)} \\
  u(x,0) &= g(x)
\end{align*}
\]

(Inhomogeneous transport equation with \( b = -1 \) and \( f(x,t) = h(x - t) - g'(x - t) \)), which gives:

\[
\begin{align*}
  u(x,t) &= g(x - tb) + \int_0^t f(x + (s - t)b, s)ds \\
  &= g(x + t) + \int_0^t f(x + t - s, s)ds \\
  &= g(x + t) + \int_0^t h(x + t - s - s) - g'(x + t - s - s)ds \quad \text{(Using def of } f) \\
  &= g(x + t) + \int_0^t h(x + t - 2s) - g'(x + t - 2s)ds
\end{align*}
\]
\[ u(x,t) = g(x+t) + \int_{x-t}^{x+t-2t} h(s') - g'(s') \left( -\frac{1}{2} ds' \right) \]

(Change of vars \( s' = x + t - 2s \))

\[ = g(x + t) - \frac{1}{2} \int_{x+t}^{x-t} h(s) - g'(s) ds \]

\[ = g(x + t) + \frac{1}{2} \int_{x-t}^{x+t} h(s) - g'(s) ds \]

\[ = g(x + t) + \frac{1}{2} \int_{x-t}^{x+t} h(s) ds - \frac{1}{2} \int_{x-t}^{x+t} g'(s) ds \]

\[ = g(x + t) + \frac{1}{2} \int_{x-t}^{x+t} h(s) ds - \frac{1}{2} \int_{x-t}^{x+t} g'(s) ds 
   - \frac{1}{2} \int_{x-t}^{x+t} g'(s) ds \]

\[ = \frac{1}{2} (g(x - t) + g(x + t)) + \int_{x-t}^{x+t} h(s) ds \]

Which, last but not least, gives the celebrated:

### D’Alembert’s Formula

The solution of the wave equation in 1 dimensions is

\[
\begin{align*}
    u_{tt} &= u_{xx} \\
    u(x,0) &= g(x) \\
    u_t(x,0) &= h(x)
\end{align*}
\]

\[ u(x, t) = \frac{1}{2} (g(x - t) + g(x + t)) + \frac{1}{2} \int_{x-t}^{x+t} h(s) ds \]

### 4. Some Consequences

Let’s look at
u(x, t) = \frac{1}{2} (g(x - t) + g(x + t)) + \frac{1}{2} \int_{x-t}^{x+t} h(s)ds

a bit more.

(1) If h \equiv 0, then we get

u(x, t) = \frac{1}{2} (g(x + t) + g(x - t))

Which means that, if there’s no initial velocity, the initial wave splits up into two half-waves, one moving to the right and the other one moving to the left.

Note: Check out the following really cool web applet that allows you to simulate solutions of the wave equation by specifying g and h: Wave Equation Simulation

(2) Note that u(x, t) depends only on the values of g and h on [x - t, x + t]. Values of g and h outside of [x - t, x + t] don’t affect u at all! This interval is sometimes called the domain of dependence. Think of the domain of dependence as a kind of
a bunker or safe haven. As long as you’re inside of the bunker, nothing in the outside world will affect you.

**Corollary:**

The wave equation has **finite** speed of propagation

More precisely, \( g(x_0) > 0 \) for some \( x_0 \) but \( g \equiv 0 \) inside \([x-t, x+t]\), then \( u(x, t) = 0 \)

This is **very** different from the heat equation, where, as we have seen, if \( g(x_0) > 0 \) somewhere, then \( u(x, t) > 0 \) everywhere!

**Analogy:** If an alien (lightyears) away lights a match, then you *immediately* feel the effect of the heat. But if that alien makes a sound, then it will take some time until you hear it (for \( t \) so large until \( x_0 \) is in \([x - t, x + t]\))

(4) There is no maximum principle for the wave equation; in general \( \max u(x, t) \neq \max g \). In other words, your wave \( u(x, t) \) could become *bigger* than your initial wave \( g(x) \) (think for instance
what happens during resonance).

Or, for example, take \( g \equiv 0 \) and \( h > 0 \), then \( u(x, t) > 0 \) but \( \max g \equiv 0 \)

(5) **Smoothness:** Usually \( u \) is not infinitely differentiable. \( u \) is generally as smooth as \( g \), and 1 degree smoother than \( h \).

For example, if \( g(x) = |x| \) (not differentiable) and \( h \equiv 0 \), then \( u(x, t) = \frac{1}{2} (|x - t| + |x + t|) \), which is also not differentiable.

(6) **Uniqueness:** Generally yes, but need to do it with energy methods since there’s no maximum principle.

(7) **Reflection Method:** (Optional) If you want to solve the wave equation on the half-line, where this time \( x > 0 \) (instead of \( x \in \mathbb{R} \)) then you can use a reflection method. See page 69 of the book, or this video: Reflection of Waves, or pages 3-9 of the following lecture notes Reflection Method. The physical phenomenon is quite interesting, where your wave just reflects off a wall. Feel free to check it out.

5. **Euler-Poisson-Darboux Equation**

**Reading:** Section 2.4.1b: Spherical Means
Of course, you may wonder: Is there a mean-value formula for the wave equation? Well yes, but actually no! There isn’t a mean-value formula here, but actually a mean-value PDE called the Euler-Poisson-Darboux equation! This will actually help us next time to solve the wave equation in 3 dimensions.

(Carefully note: If a theorem is named after a mathematician (like Fermat’s Last Theorem), then it’s important. Here it’s named after THREE mathematician, so it’s VERY important)

**Fix** $x$ and let

$$
\phi(r, t) = \int_{\partial B(x, r)} u(y, t) dS(y)
$$

**Note:** Technically, $\phi$ should also depend on $x$, but here $x$ will be constant throughout.
Claim:

\( \phi \) solves the following PDE, called the Euler-Poisson-Darboux Equation:

\[
\phi_{tt} - \phi_{rr} - \left( \frac{n-1}{r} \right) \phi_r = 0
\]

With

\[
\phi(r, 0) = \int_{\partial B(x,r)} g(y) dS(y) =: G(r)
\]

\[
\phi_t(r, 0) = \int_{\partial B(x,r)} h(y) dS(y) =: H(y)
\]

Note: Compare this to back in section 2.2 when we tried to find the fundamental solution of Laplace’s equation, then we found an expression of the form \( w'' + \left( \frac{n-1}{r} \right) w' \). In fact, the \( \phi_{rr} + \left( \frac{n-1}{r} \right) \phi_r \) term is the radial part of Laplace’s equation in polar coordinates, so the above is a sort of a wave equation (and we’ll be able to transform it to an actual wave equation next time).

Proof: Similar to the derivation of Laplace’s mean value formula!

Note: The initial conditions \( \phi(r, 0) = G(r) \) and \( \phi_t(r, 0) = H(r) \) are easy to check from the definition, so let’s just focus on the PDE.

STEP 1: Just like for Laplace’s equation, let’s change variables:
\[ \phi = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} u(y,t) dS(y) \]
\[ = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,1)} u(x + rz,t) r^{n-1} dS(z) \]

(Here we used \( z = \frac{y - x}{r} \))

\[ \phi = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x + rz,t) dS(z) \]

Therefore

\[ \phi_r = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} Du(x + rz,t) \frac{\partial}{\partial z} dS(z) \]
\[ = \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} Du(y,t) \cdot \left( \frac{y - x}{r} \right) \frac{1}{r^{n-1}} dS(y) \]
(Here we used \( y = x + rz \))
\[ = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu} dS(z) \]
\[ = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u dy \]
\[ = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} u_{tt} dy \]
(By our PDE)

**STEP 2:** Therefore, we get:
\[
\phi_r = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} u_{tt} dy
\]

\[
r^{n-1}\phi_r = \frac{1}{n\alpha(n)} \int_{B(x,r)} u_{tt} dy
\]

\[
(r^{n-1}\phi_r)_r = \frac{1}{n\alpha(n)} \left( \int_{B(x,r)} u_{tt} dy \right)_r
\]

\[
= \frac{1}{n\alpha(n)} \left( \int_0^r \int_{\partial B(x,s)} u_{tt} dS(y) \right)_r
\]

\[
= \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt} dS(y)
\]

\[
= r^{n-1} \left( \frac{\int_{\partial B(x,r)} u_{tt} dS(y)}{n\alpha(n)r^{n-1}} \right)
\]

\[
= r^{n-1} \int_{\partial B(x,r)} u_{tt} dS(y)
\]

\[
= r^{n-1} \left( \int_{\partial B(x,r)} u dS(y) \right)_{tt}
\]

\[
= r^{n-1} \phi_{tt}
\]

**STEP 3:** Hence, we get

\[
(r^{n-1}\phi_r)_r = r^{n-1}\phi_{tt}
\]

\[
(n-1)r^{n-2}\phi_r + r^{n-1}\phi_{rr} = r^{n-1}\phi_{tt}
\]

\[
(n-1)\phi_r + r\phi_{rr} = r\phi_{tt}
\]

\[
\phi_{tt} = \left( \frac{n-1}{r} \right) \phi_r + \phi_{rr}
\]

And therefore, we obtain
\[ \phi_{tt} = \phi_{rr} + \left( \frac{n-1}{r} \right) \phi_r \]