Today: All about fun applications of the Intermediate Value Theorem

Recall: Intermediate Value Theorem (IVT):

If $f : [a, b] \to \mathbb{R}$ is continuous and if $c$ is between $f(a)$ and $f(b)$, then there is some $x \in [a, b]$ with $f(x) = c$
1. **Application 1: Fixed Points**

**Video:** What is a fixed point?

**Application 1:**

If \( f : [0, 1] \to [0, 1] \) is continuous, then \( f \) has a **fixed point**: there is some \( x_0 \) in \([0, 1]\) such that \( f(x_0) = x_0 \)

(In other words, there is a point \( x_0 \) such that, when you apply \( f \) to it, then nothing happens)

Geometrically, this means that, \( f \) must cross the line \( y = x \).

**Proof:** Let \( g(x) = f(x) - x \). Then, since \( f \) is continuous, \( g \) is continuous.

Moreover, \( g(0) = f(0) - 0 = f(0) \geq 0 \)

And \( g(1) = f(1) - 1 \leq 0 \) since \( f(x) \leq 1 \) for all \( x \).
Therefore, by the IVT with \( c = 0 \) there is \( x_0 \) in \([0, 1]\) such that

\[
g(x_0) = 0 \Rightarrow f(x_0) - x_0 = 0 \Rightarrow f(x_0) = x_0 \quad \square
\]

**Applications:** Fixed points are very important in math and beyond. Here are some fun applications of fixed points:

1. No matter how well you shake a snowglobe, then there will always be one snowflake which lands on exactly same position it started!

2. Suppose there is a hurricane in New York, and everyone gets swept to a different place. Then there is one lucky person who gets swept to the same place they started!

3. Take an ordinary map of a country, and suppose that that map is laid out on a table inside that country. There will always be
a ‘You are Here’ point on the map which represents that same point in the country.

2. (Optional) Application 2: Square Roots

(Optional) Video: What is a square root?

Note: Even though this is optional, you are still required to know Example 2 in the book.

As a second application, let’s show that $\sqrt{x} = x^{\frac{1}{2}}$ exists (or, more generally, that $x^{\frac{1}{m}}$ exists).

Application 2:
If $y \geq 0$, then there is a unique $x \geq 0$ with $y = x^2$
Note: The $x$ as above is called the square root of $y$, $x = \sqrt{y}$

Proof:

Existence: Let $y \geq 0$ be given and consider $f(x) = x^2$, which is continuous. Then, on the one hand, $f(0) = 0 \leq y$, and, on the other hand:

Claim: There is $b$ such that $f(b) \geq y$

Then, since $f(0) \leq y \leq f(b)$, by the IVT, there is $x \geq 0$ such that $f(x) = y$, that is $x^2 = y$.

Proof of Claim:

Case 1: $y \leq 1$, then let $b = 1$, and we get

$$f(b) = b^2 = 1^1 = 1 \geq y$$
Case 2: $y > 1$, then let $b = y$, and we get

$$f(b) = b^2 = y^2 > y\sqrt{\text{ }}$$

Uniqueness: Suppose $a, b \geq 0$ are such that $a^2 = y$ and $b^2 = y$. Then, either $a < b$, $a > b$, or $a = b$. But if $a < b$, then we get $a^2 < b^2$, so $y < y \Rightarrow \Leftarrow$. Similar contradiction if $a > b$, and therefore we get $a = b \sqrt{\text{ }}$

Note: For a more elementary proof that doesn’t use the IVT, check out AP6 in HW2, or this video: [Construction of $x^{\frac{1}{n}}$]

3. Continuous Functions are Monotonic

Video: [Continuity and Monotonicity]
In this section, we’ll prove something truly amazing about continuous functions. Namely, one-to-one continuous functions must be either increasing or decreasing!

Recall:

\[ f \text{ is one-to-one if and only if } \quad x \neq y \Rightarrow f(x) \neq f(y) \]

(Different inputs give you different outputs)

Definition:

\( f \) is (strictly) **increasing** if \( a < b \Rightarrow f(a) < f(b) \)

\( f \) is (strictly) **decreasing** if \( a < b \Rightarrow f(a) > f(b) \)

In either case, \( f \) is (strictly) **monotonic**
Fact:

If \( f : I \to \mathbb{R} \) is one-to-one and continuous, then \( f \) must be monotonic (Recall that \( I \) is an interval)

*Intuitively,* this makes sense: Suppose \( f \) is not monotonic. Then, \( f \) goes up and down (or down and up) and cannot be one-to-one.
Example:

Let \( f(x) = \sin(x) \) on \([0, \frac{\pi}{2}]\). It can be shown that \( f \) is continuous and one-to-one. Therefore, by the fact above, \( f \) is either increasing or decreasing.

However, since \( f(0) = 0 \) and \( f\left(\frac{\pi}{2}\right) = 1 \), we must have \( f(0) < f\left(\frac{\pi}{2}\right) \), hence \( f \) cannot be decreasing, so it must be increasing on \([0, \frac{\pi}{2}]\).

Notice that, just by testing \( f \) at two points 0 and \( \frac{\pi}{2} \), we were able to figure out whether \( f \) is increasing or decreasing! (WOW) This is what makes the theorem so powerful!

**Proof:** Suppose \( f \) is continuous and one-to-one.

**STEP 1:**

**Claim:** For all \( a < b < c \)

\( f(a) < f(b) < f(c) \) or \( f(a) > f(b) > f(c) \)
Suppose not, then for some \( a < b < c \) we have

(1) \( f(b) \geq f(a) \) and \( f(b) \geq f(c) \), or

(2) \( f(b) \leq f(a) \) and \( f(b) \leq f(c) \)

(The picture illustrates the cases where \( f(c) > f(a) \), but the cases where \( f(c) < f(a) \) are similar)
WLOG, assume (1), that is \( f(b) \geq f(a) \) and \( f(b) \geq f(c) \) (the other case is similar).

Since \( f \) is one-to-one, we have \( f(b) \neq f(a) \) and \( f(b) \neq f(c) \), hence (1) becomes \( f(b) > f(a) \) and \( f(b) > f(c) \).

Let \( y \) be a number that is both (strictly) between \( f(a) \) and \( f(b) \) and between \( f(b) \) and \( f(c) \).

Since \( f \) is continuous, by the IVT on \( (a,b) \), there must be \( x_1 \) in \( (a,b) \) such that \( f(x_1) = y \). And by the IVT on \( (b,c) \) there must be \( x_2 \) in \( (b,c) \) with \( f(x_2) = y \).

But then \( f(x_1) = f(x_2) = y \) whereas \( x_1 \neq x_2 \), which contradicts \( f \) being one-to-one \( \Rightarrow \Leftrightarrow \checkmark \).

**STEP 2:** Therefore, for all \( a < b < c \), either \( f(a) < f(b) < f(c) \) or \( f(a) > f(b) > f(c) \).

**Problem:** In theory have a function \( f \), we have \( f(a) < f(b) < f(c) \) for some \( a < b < c \), and \( f(a) > f(b) > f(c) \) for other \( a < b < c \), which is not monotonic, as in the following picture:
Essentially, we need to rule out one of the two possibilities.

Fix $a_0 < b_0$ in $I$ (Think of $a_0$ and $b_0$ as helper numbers because they help us determine if $f$ is increasing or decreasing. In the $\sin(x)$ example above, $a_0 = 0$ and $b_0 = \frac{\pi}{2}$)

Since $f$ is one-to-one, we have $f(a_0) \neq f(b_0)$, hence either $f(a_0) < f(b_0)$ or $f(a_0) > f(b_0)$.

Assume WLOG $f(a_0) < f(b_0)$ (the other case is similar but would give you $f$ decreasing)
Goal: Show $f$ is increasing.

STEP 3: Let $x \in I$

Claim:

$x < a_0 \Rightarrow f(x) < f(a_0)$

$x > a_0 \Rightarrow f(x) > f(a_0)$

(This is not quite the same as $f$ being increasing, since $a_0$ is fixed here)

Case 1: $x < a_0$
Then, since \( x < a_0 < b_0 \) and \( f(a_0) < f(b_0) \), by STEP 1, we must have \( f(x) < f(a_0) < f(b_0) \) so \( f(x) < f(a_0) \) ✓

**Case 2:** \( x > a_0 \)
Case 2a: If \( a_0 < x < b_0 \), then, similar to Case 1, we get \( f(a_0) < f(x) < f(b_0) \) so \( f(x) > f(a_0) \) ✓

Case 2b: If \( x = b_0 \), then we get \( f(x) = f(b_0) > f(a_0) \) ✓

Case 2c: If \( x > b_0 \), then since \( a_0 < b_0 < x \) and therefore \( f(a_0) < f(b_0) < f(x) \), so \( f(x) > f(a_0) \) ✓

Therefore we get \( f(x) > f(a_0) \) □

STEP 4:

Claim: \( f \) is increasing

Suppose \( x_1 < x_2 \) and show \( f(x_1) < f(x_2) \)

Case 1: \( x_1 < x_2 < a_0 \)
Since \( x_2 < a_0 \) then STEP 3 implies \( f(x_2) < f(a_0) \), and therefore from STEP 1, have \( f(x_1) < f(x_2) < f(a_0) \), and hence \( f(x_1) < f(x_2) \) \( \checkmark \)

**Case 2:** \( x_1 \leq a_0 \leq x_2 \)

Since \( x_1 \leq a_0 \), we get \( f(x_1) \leq f(a_0) \), and since \( x_2 \geq a_0 \) we get \( f(x_2) \geq f(a_0) \), and therefore \( f(x_1) \leq f(a_0) \leq f(x_2) \), hence \( f(x_1) \leq f(x_2) \). Moreover, since \( x_1 \neq x_2 \) and \( f \) is one-to-one we have \( f(x_1) \neq f(x_2) \). Hence \( f(x_1) < f(x_2) \) \( \checkmark \)

**Case 3:** \( a_0 < x_1 < x_2 \).

Since \( a_0 < x_1 \) we get \( f(a_0) < f(x_1) \) from STEP 3, and therefore, since \( a_0 < x_1 < x_2 \), we get \( f(a_0) < f(x_1) < f(x_2) \) and hence \( f(x_1) < f(x_2) \) \( \checkmark \)

In either case, we get that \( f \) is increasing \( \square \)

4. \( f^{-1} \) IS CONTINUOUS

**Video:** \( f^{-1} \) is continuous

Finally, let’s prove the incredible fact that if a real-valued \( f \) is continuous, then \( f^{-1} \) is continuous as well. This explains the fact why \( \sqrt{x}, \tan^{-1}(x) \), or even \( \ln(x) \) are continuous.

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\(^1\)If \( x_1 < a_0 \) then \( f(x_1) < f(a_0) \) and if \( x_1 = a_0 \) we get \( f(x_1) = f(a_0) \) in which case the inequality holds.
Recall: (Math 13)

(1) The **image** of an interval $I$ is

$$f(I) = \{ f(x) \mid x \in I \}$$

(2) If $f : I \to f(I)$ is one-to-one, $f^{-1} : f(I) \to I$ is defined by

$$f(x) = y \Leftrightarrow f^{-1}(y) = x$$

In other words, if $f$ as a flight from $x$ to $y$, then $f^{-1}$ is the return flight from $y$ to $x$. $f^{-1}$ undoes whatever $f$ does.

Before we show that $f^{-1}$ is continuous, let’s first show a fact that will be useful at the end.
Lemma:
If \( f : I \to f(I) \) is increasing then \( f^{-1} : f(I) \to I \) is also increasing

Proof: Suppose \( y_1, y_2 \in f(I) \) are such that \( y_1 < y_2 \). We need to show \( f^{-1}(y_1) < f^{-1}(y_2) \)

By definition of \( f(I) \), there is \( x_1 \in I \) with \( y_1 = f(x_1) \) and there is \( x_2 \in I \) with \( y_2 = f(x_2) \). In particular \( x_1 = f^{-1}(y_1) \) and \( x_2 = f^{-1}(y_2) \)

Now if \( x_1 \geq x_2 \), since \( f \) is increasing, we would have \( f(x_1) \geq f(x_2) \) and therefore, by definition \( y_1 \geq y_2 \Rightarrow \)

Therefore we must have \( x_1 < x_2 \), that is \( f^{-1}(y_1) < f^{-1}(y_2) \).

We have shown that \( y_1 < y_2 \Rightarrow f^{-1}(y_1) < f^{-1}(y_2) \), and therefore \( f^{-1} \) is increasing \( \square \)
**Theorem:**

If $f : I \to f(I)$ is one-to-one and continuous, then $f^{-1} : f(I) \to I$ is continuous as well.

**Proof:** Since $f$ is continuous, from the previous section, $f$ is either increasing or decreasing, so WLOG, assume $f$ is increasing.

To simplify notation, let $J = f(I)$ and $g = f^{-1}$

**Goal:** Prove that for all $x_0 \in J$, $g$ is continuous at $x_0$

**STEP 1:** First of all, since $I$ is an interval and $f$ is continuous, then $J = f(I)$ is also interval (from last time)

For simplicity, assume that $x_0$ is not an endpoint of $J$ (for example if $J = [2, 3]$, assume $x_0$ is neither 2 or 3). The general case is similar.

**Claim:** $g(x_0)$ is not an endpoint of $I$

**Proof of Claim:** Suppose not, and assume for example that $g(x_0)$ is the left endpoint of $I$.

Let $x \in J$ be arbitrary. Then $g(x) \in I$, and, since $g(x_0)$ is the left endpoint of $I$, we must have $g(x_0) \leq g(x)$
But then, since $f$ is increasing and $g = f^{-1}$, we get

$$f (g(x_0)) \leq f (g(x)) \Rightarrow x_0 \leq x$$

But then this means that $x_0$ is the left endpoint of $J$, which is a contradiction $\Rightarrow \Leftarrow$

Since $g(x_0)$ is not an endpoint of $I$, it must be in the interior of $I$, and so there exists $r > 0$ such that $(g(x_0) - r, g(x_0) + r) \subseteq I$
**STEP 2:** Let $\epsilon > 0$ be given, and assume $\epsilon$ so small that $\epsilon < r$

Then, since $\epsilon < r$ and by STEP 1, we have:

$$[g(x_0) - \epsilon, g(x_0) + \epsilon] \subseteq (g(x_0) - r, g(x_0) + r) \subseteq I$$

However, since $g(x_0) - \epsilon$ and $g(x_0) + \epsilon$ are in $I$ and $g : J \to I$ is onto (since $g$ is invertible), there are $x_1$ and $x_2$ in $J$ such that $g(x_0) - \epsilon = g(x_1)$ and $g(x_0) + \epsilon = g(x_2)$
Claim: $x_1 < x_0 < x_2$

(This is at least true from the picture)

This is because

\[
g(x_0) - \epsilon < g(x_0) < g(x_0) + \epsilon
\]
\[
\Rightarrow g(x_1) < g(x_0) < g(x_2)
\]
\[
\Rightarrow f(g(x_1)) < f(g(x_0)) < f(g(x_2)) \quad \text{(Since } f \text{ is increasing)}
\]
\[
\Rightarrow x_1 < x_0 < x_2 \quad \text{(Since } g = f^{-1})
\]

**STEP 3: Actual Proof**

With $\epsilon$ as above, let $\delta = \min \left\{ |x_2 - x_0|, |x_1 - x_0| \right\}$
**Intuition:** With this $\delta$, any $x$ that is $\delta$—close to $x_0$ is guaranteed to be in the blue region above, and therefore $g(x)$ is guaranteed to be in the red/good region, so $g(x)$ will be $\epsilon$—close to $g(x_0)$

Then if $|x - x_0| < \delta$, then by definition of $\delta$, we get $x_1 < x < x_2$ and therefore

\[
x_1 < x < x_2
\]
\[
\implies g(x_1) < g(x) < g(x_2) \quad \text{(Since $g$ is increasing)}
\]
\[
\implies g(x_0) - \epsilon < g(x) < g(x_0) + \epsilon \quad \text{(By definition of $x_1$ and $x_2$)}
\]
\[
\implies -\epsilon < g(x) - g(x_0) < \epsilon
\]
\[
\implies |g(x) - g(x_0)| < \epsilon
\]

Therefore $g = f^{-1}$ is continuous at $x_0$ \qedsymbol