LECTURE 25: PROPERTIES OF CONTINUOUS FUNCTIONS (I)

Today: We'll prove two of the three *Value Theorems* used in Calculus: The Extreme Value Theorem and the Intermediate Value Theorem. The Mean Value Theorem will be proven in Math 140B.

Note: To give you a break, today we will not use any ϵ and δ ☺

1. Bounded Functions

Video: Bounded Functions

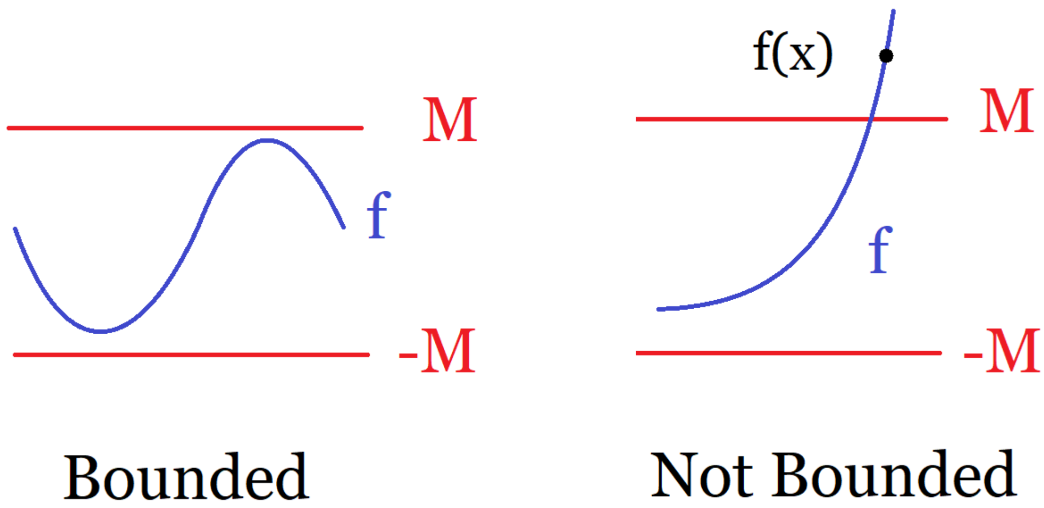
As a warm-up, let's show that continuous functions are bounded

Defintion:

f is **bounded** if there is $M > 0$ such that for all x, $|f(x)| \leq M$

(This is similar to the definition of sequences being bounded)

Date: Wednesday, May 27, 2020.

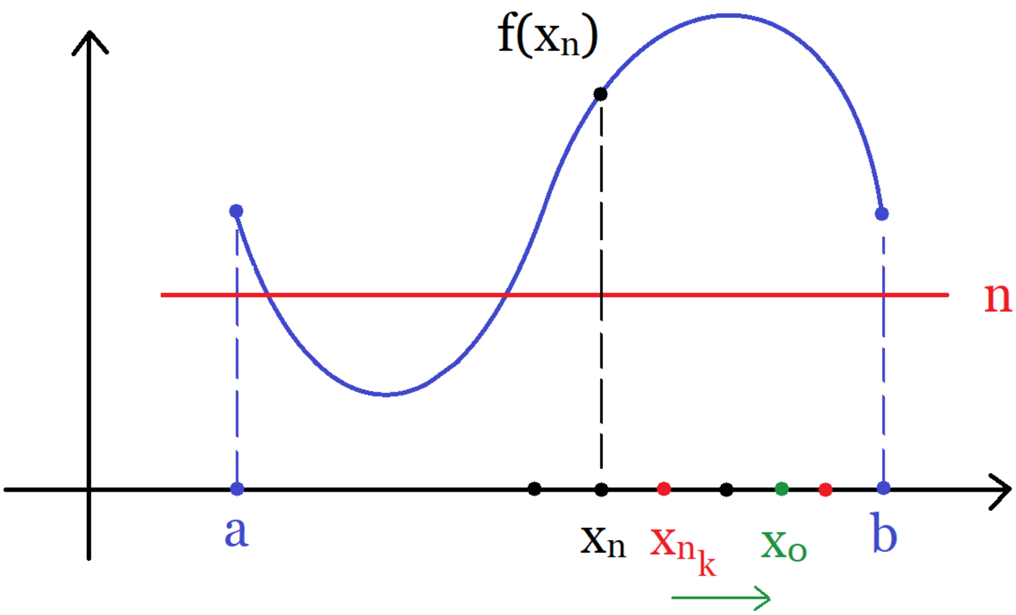


In other words, a bounded function is trapped between $-M$ and M, whereas an unbounded function always goes outside of $[-M, M]$, no matter how large M is.

Fact:

If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded

Proof: Suppose not. Then for all $n \in \mathbb{N}$ (using the above with $M = n$) there is some $x_n \in [a, b]$ with $|f(x_n)| > n$.



Since $x_n \in [a, b]$, the sequence (x_n) is bounded.

Therefore, by Bolzano-Weierstraß , (x_n) has a convergent subsequence (x_{n_k}) that converges to some $x_0 \in [a, b]$.

Since $x_{n_k} \to x_0$ and f is continuous, we have $f(x_{n_k}) \to f(x_0)$ and so $|f(x_{n_k})| \to |f(x_0)|$

On the other hand, since $|f(x_n)| > n$ for all n, we have $|f(x_n)| \to \infty$. In particular this is true for the subsequence $f(x_{n_k})$, and therefore $|f(x_{n_k})| \to \infty$ as well.

Comparing the two, we get $|f(x_0)| = \infty$, which is absurd $\Rightarrow\Leftarrow$ $\square$

2. The Extreme Value Theorem

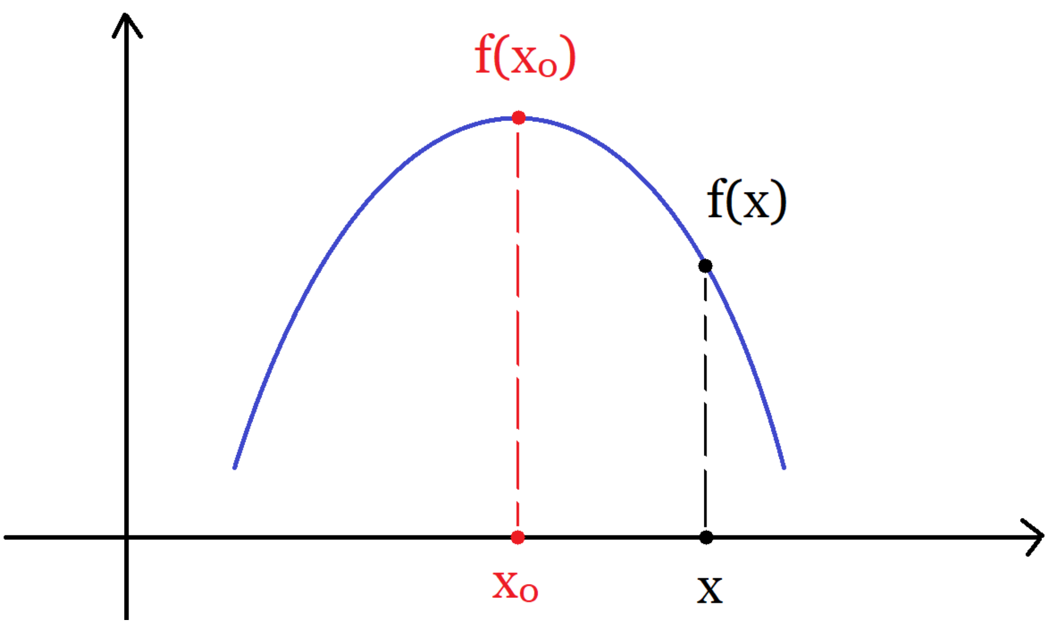
Video: The Extreme Value Theorem

The Extreme Value Theorem is of the unsung heroes in Calculus. It says that any continuous function f on $[a, b]$ must have a maximum and minimum. Without this, optimization problems would be impossible to solve!

Defintion:

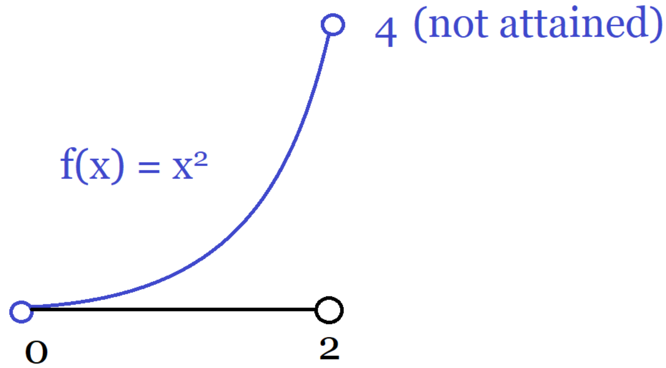
f has a **maximum** on $[a, b]$ if there is $x_0 \in [a, b]$ such that $f(x_0) \geq f(x)$ for all $x \in [a, b]$

(Similarly for minimum)



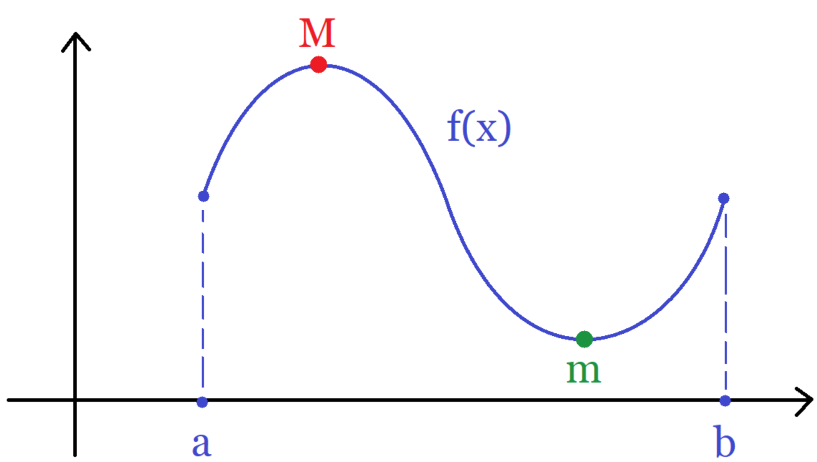
Important: *By definition*, the maximum has to be attained. In other words, there must be some x_0 such that $f(x_0)$ is that maximum!

Non-Example: $f(x) = x^2$ has no maximum on $(0, 2)$ because if it did, the maximum would be 4, but there is no x_0 in $(0, 2)$ with $f(x_0) = 4$



Extreme Value Theorem:

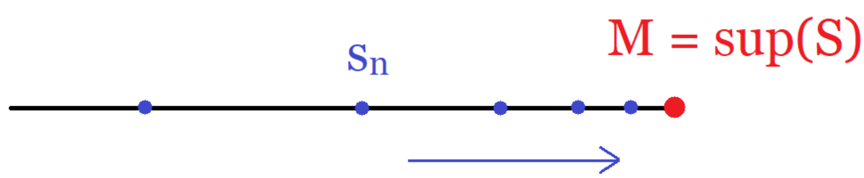
Suppose $f : [a, b] \to \mathbb{R}$ is continuous, then f has a maximum and a minimum on $[a, b]$



First, let's prove a Useful Lemma that will be useful both here and for the Intermediate Value Theorem:

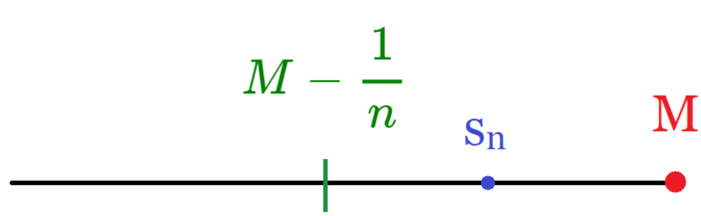
Useful Lemma:

If S is a subset of $\mathbb{R}$ with $\sup(S) =: M < \infty$, then there is a sequence (s_n) in S that converges to M



In other words, there is always a train with destination $\sup(S)$. That is, you can always reach $\sup(S)$ with a sequence. In other words, $\sup(S)$ isn't such an abstract concept any more, we can attain it through sequences!

Proof of Useful Lemma: For every $n \in \mathbb{N}$, consider $M - \frac{1}{n} < M = \sup(S)$. Therefore by definition of sup, for every n there is some $s_n \in S$ with $M - \frac{1}{n} < s_n \leq M$, but then by the squeeze theorem, we have $s_n \to M$ $\square$



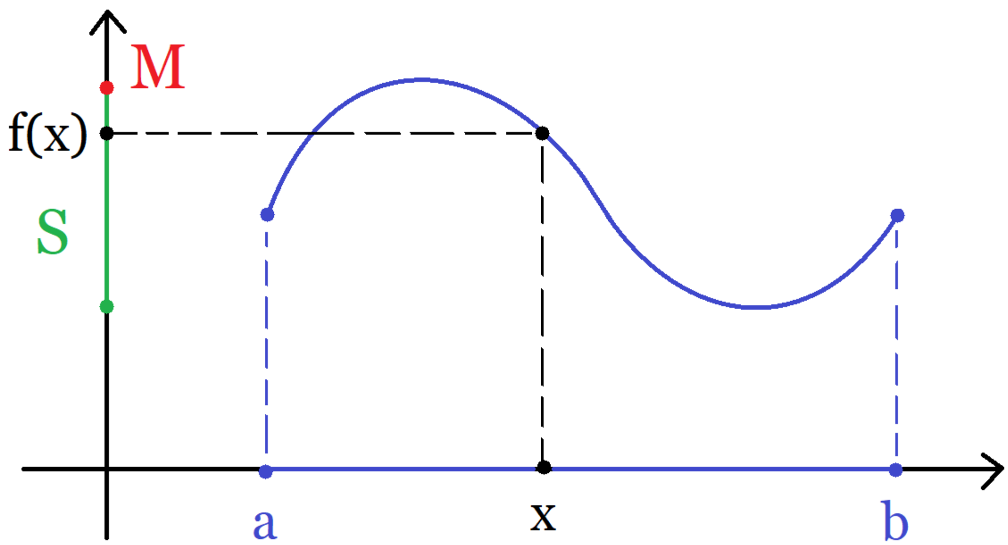
Proof of the Extreme Value Theorem:

Note: It's enough to show that f has a maximum, since we can repeat the same proof with $-f$ instead of f (since $-f$ has a maximum whenever f has a minimum, and vice-versa)

STEP 1: Since f is continuous on $[a, b]$, f is bounded, so there is C such that $|f(x)| \leq C$ for all x.

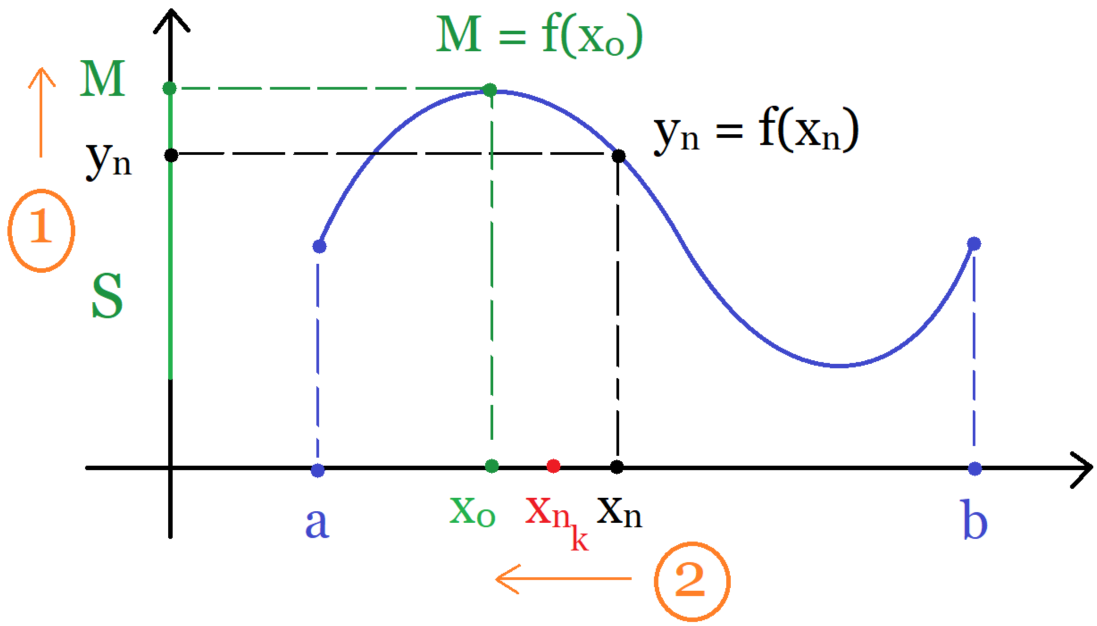
Consider the set

$$S = \{f(x) \mid x \in [a, b]\}$$



Since f is bounded, S is bounded (by C), and therefore S has a least upper bound $\sup(S) =: M$

STEP 2: By the Useful Lemma above, there is a sequence $y_n \in S$ with $y_n \to M$



Since $y_n \in S$, by definition of S we have $y_n = f(x_n)$ for some $x_n \in [a, b]$

Since $x_n \in [a, b]$, (x_n) is bounded. Therefore, by the Bolzano-Weierstraß theorem, (x_n) has a convergent subsequence $x_{n_k} \to x_0$ for some $x_0 \in [a, b]$

Since $x_{n_k} \to x_0$ and f is continuous, we have $f(x_{n_k}) \to f(x_0)$

On the other hand, $f(x_n) = y_n \to M$ (by definition of y_n). In particular, the subsequence $f(x_{n_k}) = y_{n_k} \to M$ as well.

Comparing the two, we get $f(x_0) = M$

STEP 3:

Claim: f has a maximum

This follows because for all $x \in [a, b]$

$$f(x_0) = M = \sup \{f(x) \mid x \in [a, b]\} \geq f(x)$$

And therefore $f(x_0) \geq f(x)$ for all $x \in [a, b]$ □

Note: The same result holds if you replace $[a, b]$ by any compact set. Check out this video if you're interested: Continuity and Compactness

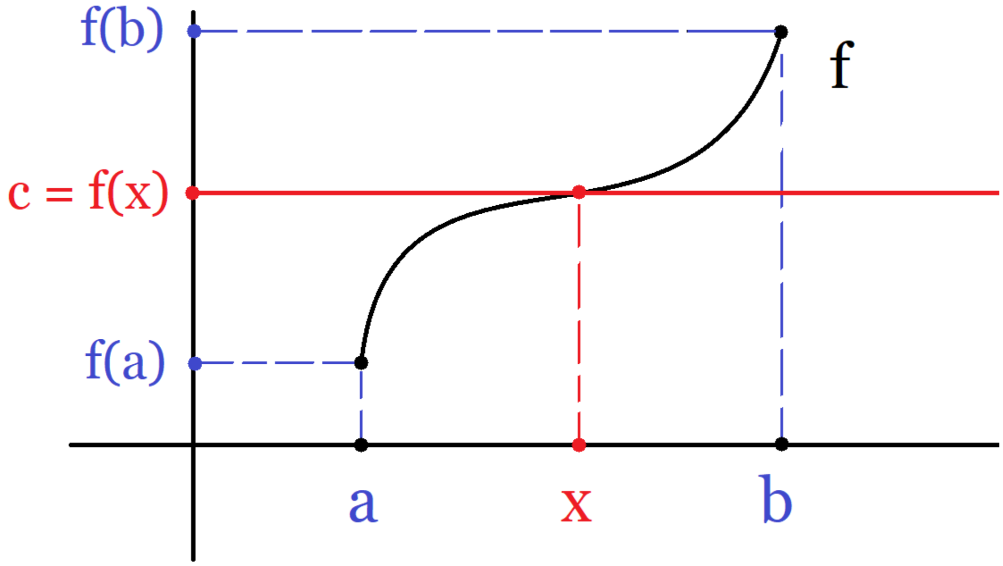
3. The Intermediate Value Theorem

Video: Intermediate Value Theorem

Let's now discuss the second Value Theorem of Calculus: The Intermediate Value Theorem. It says that if f is continuous, then f attains all the values between $f(a)$ and $f(b)$:

Intermediate Value Theorem:

If $f : [a, b] \to \mathbb{R}$ is continuous and if c is any number between $f(a)$ and $f(b)$, then there is some $x \in [a, b]$ such that $f(x) = c$

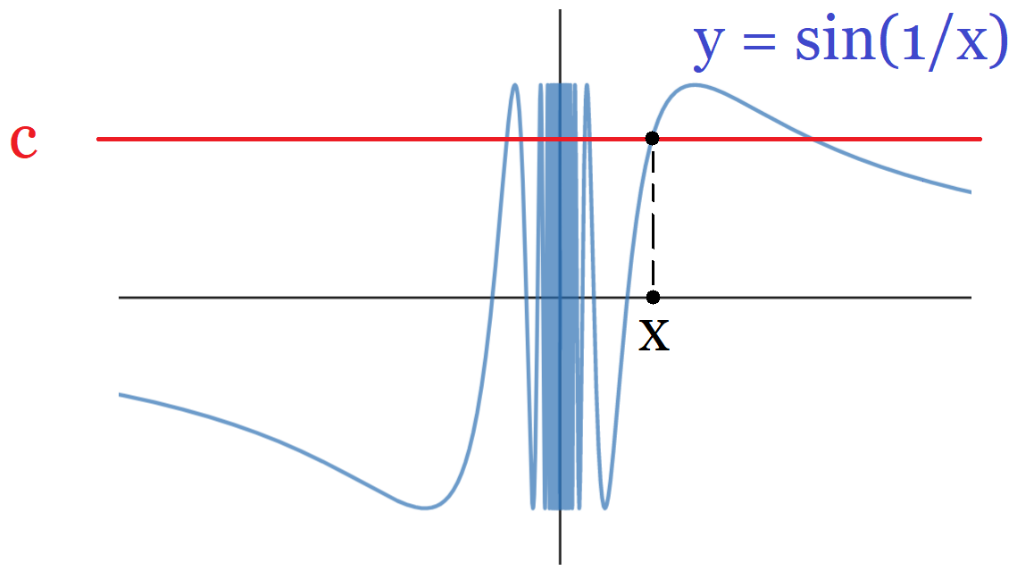


Note: There are functions f that are not continuous, but that satisfy the intermediate value property above.

Example: (see HW)

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Discontinuous at 0 but satisfies the intermediate value property

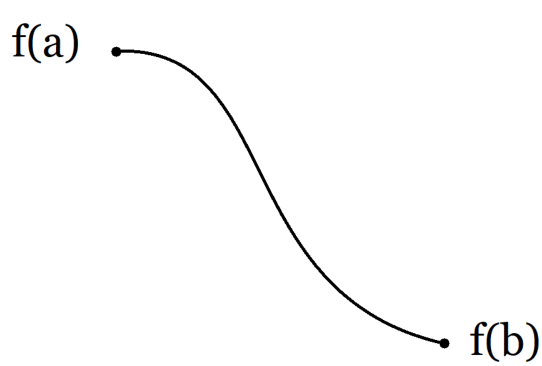


(Graph courtesy Desmos)

Proof:

STEP 1: WLOG, assume $f(a) < c < f(b)$

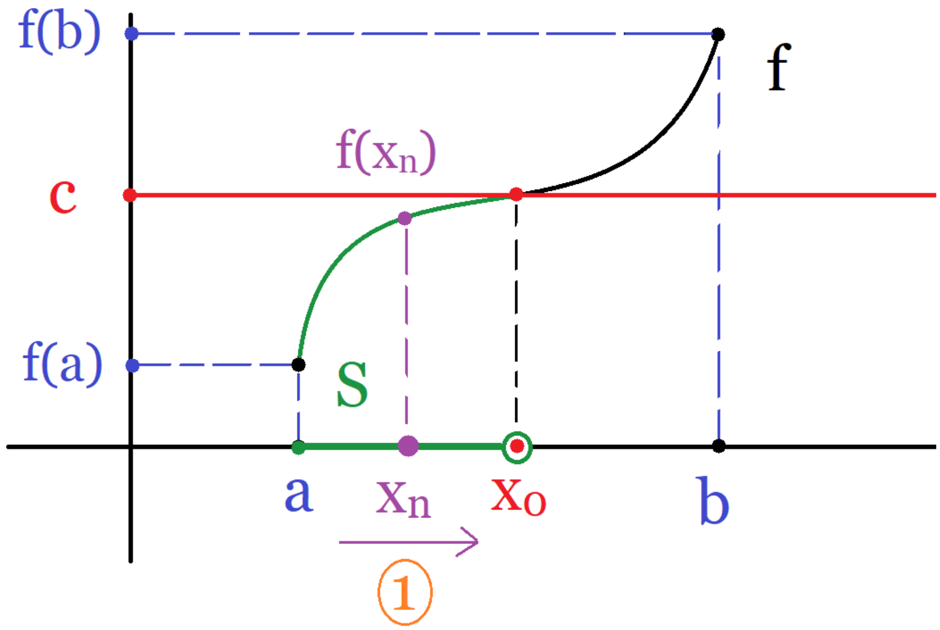
(If $c = f(a)$, let $x = a$, and if $c = f(b)$, let $x = b$. And if $f(b) < f(a)$, apply the result with $-f$)



Since $f(a) < c$, let's consider

$$S = \{x \in [a, b] \mid f(x) < c\}$$

Then $S \neq \emptyset$ (since $a \in S$) and S is bounded above (by b), therefore S has a least upper bound $\sup(S) =: x_0$



Claim: $f(x_0) = c$

We will do this by showing $f(x_0) \leq c$ and $f(x_0) \geq c$

STEP 2: Show $f(x_0) \leq c$

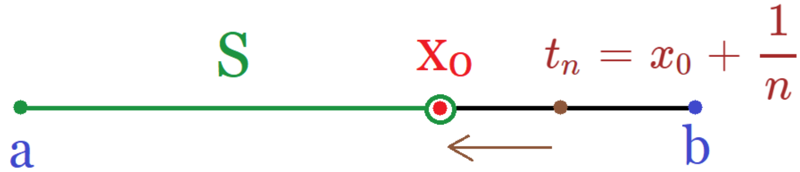
By the Useful Lemma from above, there is a sequence (x_n) in S with $x_n \to x_0$. Therefore, since f is continuous, we get $f(x_n) \to f(x_0)$.

But since $x_n \in S$, by definition of S, we have $f(x_n) < c$, and therefore

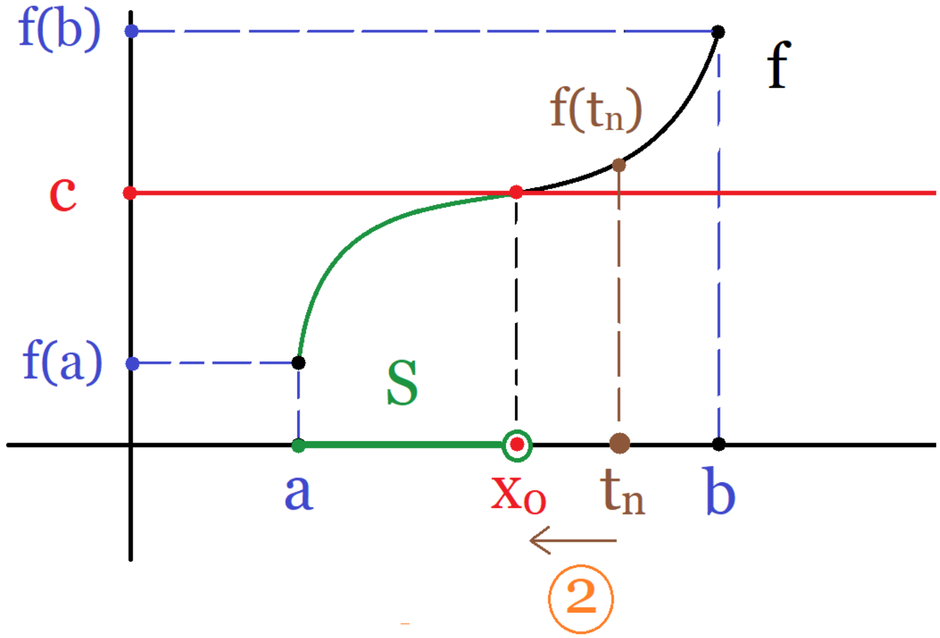
$$f(x_0) = \lim_{n\to\infty} f(x_n) \leq c \checkmark$$

STEP 3: Show $f(x_0) \geq c$

First of all, we have $x_0 \neq b$ because $f(x_0) \geq c$ whereas $f(b) > c$, so x_0 and b cannot be equal. Therefore $x_0 < b$.



Since $x_0 < b$, for n small enough, we have $t_n =: x_0 + \frac{1}{n} < b$. By definition $t_n \in [a,b]$, $t_n > x_0$ and $t_n \to x_0$.



Since $t_n \to x_0$ and f is continuous, we have $f(t_n) \to f(x_0)$

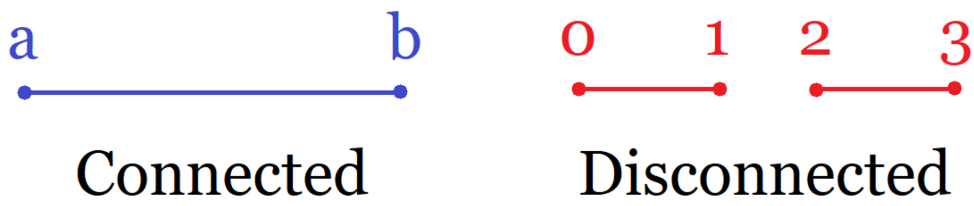
Moreover, since $t_n > x_0$ and $x_0 = \sup(S)$, we must have $t_n \notin S$, meaning that (by definition of S), $f(t_n) \geq c$.

Therefore, we get

$$f(x_0) = \lim_{n\to\infty} f(t_n) \geq c\checkmark$$

Combining STEP 2 and STEP 3, we get $f(x_0) = c$ $\square$

Note: The same result holds if you replace $[a, b]$ by any *connected* set. Connected intuitively just means that the set just had one piece. For instance $[a, b]$ is connected but $[0, 1] \cup [2, 3]$ is disconnected; it has two pieces.



4. Image of an interval

Video: Image of an interval

Because the Intermediate Value Theorem, it is interesting to figure out what happens when you apply a function to an interval.

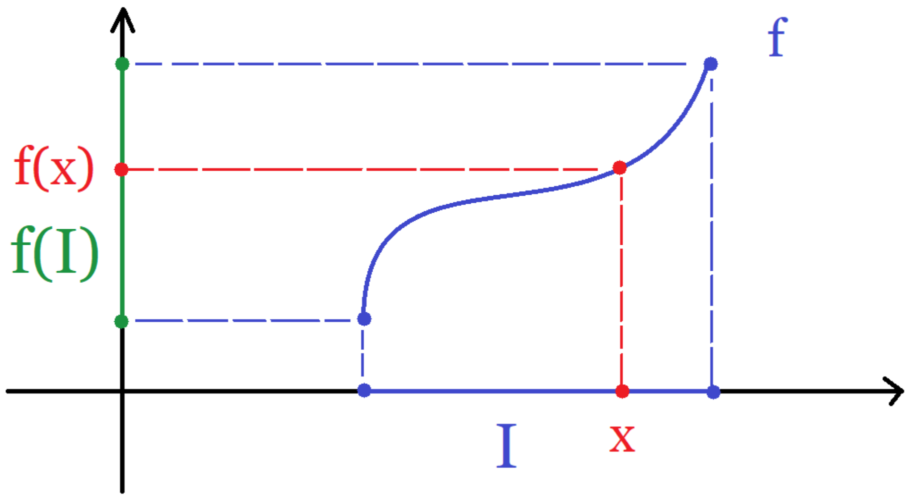
Notation:

I denotes an interval, such as $I = (0, 1)$ or $[1, 2)$ or $[2, 3]$ or $(3, \infty)$ or even $\mathbb{R}$

Definition:

If I is an interval then the **image of** f **of** I (or the range of f) is

$$f(I) = \{f(x) \mid x \in I\}$$



For general f, $f(I)$ could be some crazy set (think a fractal or the Cantor set), but it turns out that if f is continuous, then $f(I)$ is very nice:

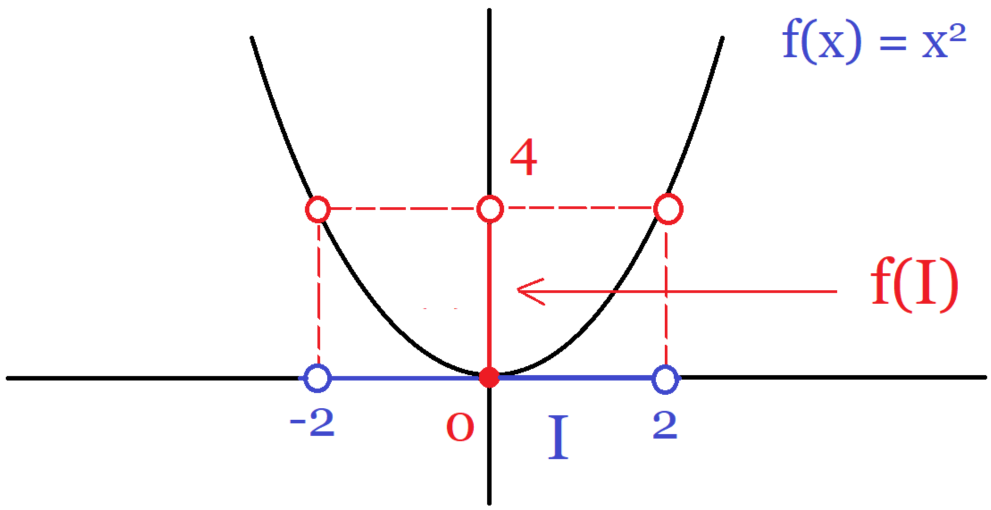
Fact:

If f is continuous, then $f(I)$ is an interval (or a single point)

Example 1:

If $f(x) = x^2$ and $I = (-2, 2)$ then

$$f(I) = \left\{x^2 \mid x \in (-2, 2)\right\} = [0, 4)$$

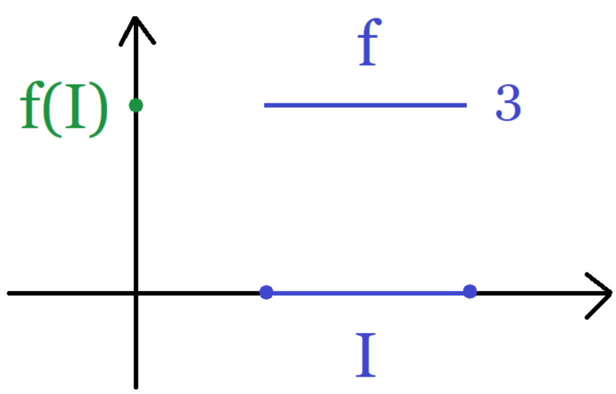


Beware: Even though $(-2, 2)$ is open, $f((-2, 2))$ isn't necessarily open!

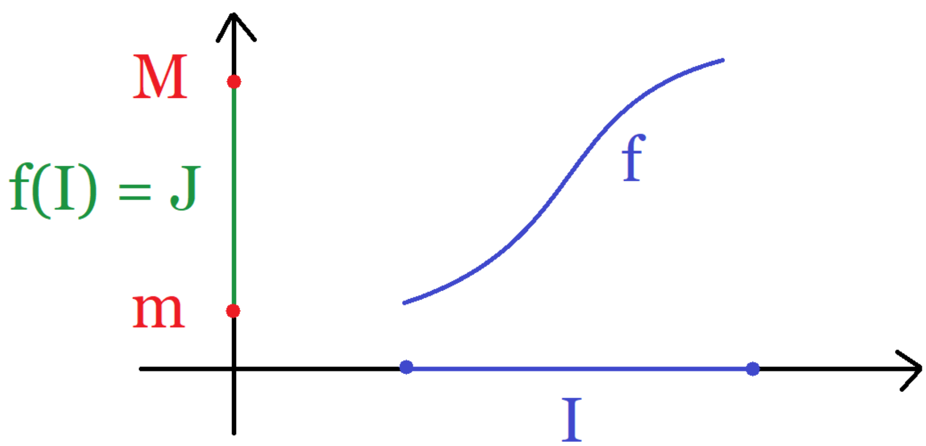
Example 2:

If $f(x) \equiv 3$ and I is any nonempty interval, then

$$f(I) = \{3\}$$



Proof: Let $J =: f(I)$ and let $m =: \inf(J)$ and $M =: \sup(J)$



Case 1: $m = M$, then $J = \{m\}$ is a single point ✓

Case 2: $m < M$.

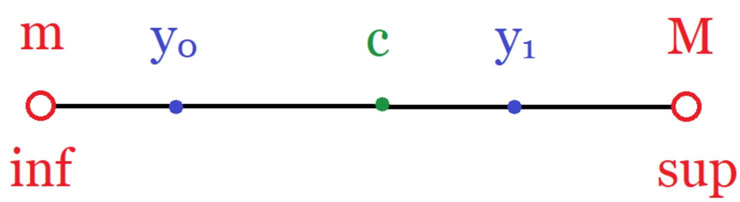
Claim: J contains the interval (m, M)

Then we would be done because we would then have either $J = (m, M)$ or $J = [m, M)$ or $J = (m, M]$ or $J = [m, M]$, depending on whether or not $m = \inf(J)$ and $M = \sup(J)$ are in J or not (here the endpoints may be infinite).

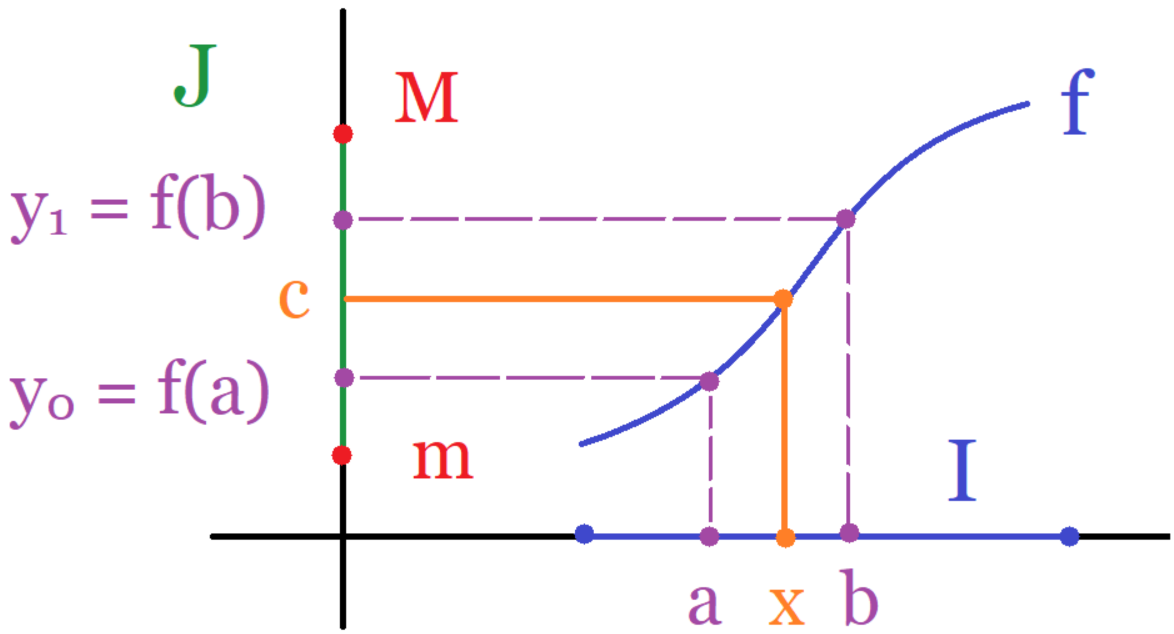
Proof of Claim: Let $c \in (m, M)$, and show $c \in J$.

By assumption $m < c < M$. Since $c > m = \inf(J)$, by definition of inf, there is $y_0 \in J$ such that $y_0 < c$, and since $c < M = \sup(J)$, there is $y_1 \in J$ such that $c < y_1$.

Therefore we get $y_0 < c < y_1$.



Since $y_0 \in J = f(I)$, by definition of $f(I)$, there is $a \in I$ such that $y_0 = f(a)$. Similarly there is $b \in I$ such that $y_1 = f(b)$.



Since f is continuous and c is between $f(a)$ and $f(b)$, by the Intermediate Value Theorem, there is x between $a \in I$ and $b \in I$ (so $x \in I$ since I is an interval) such that $f(x) = c$, but this means that $c \in f(I) = J$ ✓ □