LECTURE 25: PROPERTIES OF CONTINUOUS
FUNCTIONS (I)

Today: We’ll prove two of the three Value Theorems used in Calculus:
The Extreme Value Theorem and the Intermediate Value Theorem. The Mean Value Theorem will be proven in Math 140B.

Note: To give you a break, today we will not use any $\epsilon$ and $\delta$

1. Bounded Functions

Video: [Bounded Functions]

As a warm-up, let’s show that continuous functions are bounded

<table>
<thead>
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<th>Defintion:</th>
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<td>$f$ is bounded if there is $M &gt; 0$ such that for all $x$, $</td>
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(This is similar to the definition of sequences being bounded)

Date: Wednesday, May 27, 2020.
In other words, a bounded function is trapped between $-M$ and $M$, whereas an unbounded function always goes outside of $[-M, M]$, no matter how large $M$ is.

**Fact:**

If $f : [a, b] \to \mathbb{R}$ is continuous, then $f$ is bounded

**Proof:** Suppose not. Then for all $n \in \mathbb{N}$ (using the above with $M = n$) there is some $x_n \in [a, b]$ with $|f(x_n)| > n$. 
Since $x_n \in [a, b]$, the sequence $(x_n)$ is bounded.

Therefore, by Bolzano-Weierstraß, $(x_n)$ has a convergent subsequence $(x_{n_k})$ that converges to some $x_0 \in [a, b]$.

Since $x_{n_k} \to x_0$ and $f$ is continuous, we have $f(x_{n_k}) \to f(x_0)$ and so $|f(x_{n_k})| \to |f(x_0)|$

On the other hand, since $|f(x_n)| > n$ for all $n$, we have $|f(x_n)| \to \infty$. In particular this is true for the subsequence $f(x_{n_k})$, and therefore $|f(x_{n_k})| \to \infty$ as well.

Comparing the two, we get $|f(x_0)| = \infty$, which is absurd $\Rightarrow \Leftarrow$  

2. The Extreme Value Theorem
Video: The Extreme Value Theorem

The Extreme Value Theorem is of the unsung heroes in Calculus. It says that any continuous function $f$ on $[a, b]$ must have a maximum and minimum. Without this, optimization problems would be impossible to solve!

**Definition:**

$f$ has a **maximum** on $[a, b]$ if there is $x_0 \in [a, b]$ such that $f(x_0) \geq f(x)$ for all $x \in [a, b]$.

(Similarly for minimum)

**Important:** By definition, the maximum has to be **attained**. In other words, there must be some $x_0$ such that $f(x_0)$ is that maximum!
**Non-Example:** $f(x) = x^2$ has no maximum on $(0, 2)$ because if it did, the maximum would be 4, but there is no $x_0$ in $(0, 2)$ with $f(x_0) = 4$.

**Extreme Value Theorem:**

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then $f$ has a maximum and a minimum on $[a, b]$. 
First, let’s prove a Useful Lemma that will be useful both here and for the Intermediate Value Theorem:

**Useful Lemma:**

If $S$ is a subset of $\mathbb{R}$ with $\sup(S) = M < \infty$, then there is a sequence $(s_n)$ in $S$ that converges to $M$.

In other words, there is always a train with destination $\sup(S)$. That is, you can always reach $\sup(S)$ with a sequence. In other words, $\sup(S)$ isn’t such an abstract concept any more, we can attain it through sequences!

**Proof of Useful Lemma:** For every $n \in \mathbb{N}$, consider $M - \frac{1}{n} < M = \sup(S)$. Therefore by definition of sup, for every $n$ there is some $s_n \in S$ with $M - \frac{1}{n} < s_n \leq M$, but then by the squeeze theorem, we have $s_n \to M$. 

□
Proof of the Extreme Value Theorem:

Note: It’s enough to show that \( f \) has a maximum, since we can repeat the same proof with \(-f\) instead of \( f \) (since \(-f\) has a maximum whenever \( f \) has a minimum, and vice-versa)

STEP 1: Since \( f \) is continuous on \([a,b]\), \( f \) is bounded, so there is \( C \) such that \(|f(x)| \leq C\) for all \( x \).

Consider the set

\[
S = \{ f(x) \mid x \in [a, b] \}
\]

Since \( f \) is bounded, \( S \) is bounded (by \( C \)), and therefore \( S \) has a least upper bound \( \sup(S) =: M \)

STEP 2: By the Useful Lemma above, there is a sequence \( y_n \in S \) with \( y_n \to M \)
Since $y_n \in S$, by definition of $S$ we have $y_n = f(x_n)$ for some $x_n \in [a, b]$

Since $x_n \in [a, b], (x_n)$ is bounded. Therefore, by the Bolzano-Weierstraß theorem, $(x_n)$ has a convergent subsequence $x_{n_k} \rightarrow x_0$ for some $x_0 \in [a, b]$

Since $x_{n_k} \rightarrow x_0$ and $f$ is continuous, we have $f(x_{n_k}) \rightarrow f(x_0)$

On the other hand, $f(x_n) = y_n \rightarrow M$ (by definition of $y_n$). In particular, the subsequence $f(x_{n_k}) = y_{n_k} \rightarrow M$ as well.

Comparing the two, we get $f(x_0) = M$

**STEP 3:**
**Claim:** $f$ has a maximum

This follows because for all $x \in [a, b]$

$$f(x_0) = M = \sup \{f(x) \mid x \in [a, b]\} \geq f(x)$$

And therefore $f(x_0) \geq f(x)$ for all $x \in [a, b]$ \hfill \square

**Note:** The same result holds if you replace $[a, b]$ by any compact set. Check out this video if you’re interested: Continuity and Compactness

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3. **The Intermediate Value Theorem**

**Video:** Intermediate Value Theorem

Let’s now discuss the second Value Theorem of Calculus: The Intermediate Value Theorem. It says that if $f$ is continuous, then $f$ attains all the values between $f(a)$ and $f(b)$:

**Intermediate Value Theorem:**

If $f : [a, b] \to \mathbb{R}$ is continuous and if $c$ is any number between $f(a)$ and $f(b)$, then there is some $x \in [a, b]$ such that $f(x) = c$
Note: There are functions $f$ that are not continuous, but that satisfy the intermediate value property above.

**Example: (see HW)**

$$f(x) = \begin{cases} \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Discontinuous at 0 but satisfies the intermediate value property
Proof:

**STEP 1:** WLOG, assume \( f(a) < c < f(b) \)

(If \( c = f(a) \), let \( x = a \), and if \( c = f(b) \), let \( x = b \). And if \( f(b) < f(a) \), apply the result with \(-f\))
Since $f(a) < c$, let’s consider

$$S = \{x \in [a, b] \mid f(x) < c\}$$

Then $S \neq \emptyset$ (since $a \in S$) and $S$ is bounded above (by $b$), therefore $S$ has a least upper bound $\sup(S) =: x_0$

**Claim:** $f(x_0) = c$

We will do this by showing $f(x_0) \leq c$ and $f(x_0) \geq c$

**STEP 2:** Show $f(x_0) \leq c$

By the Useful Lemma from above, there is a sequence $(x_n)$ in $S$ with $x_n \to x_0$. Therefore, since $f$ is continuous, we get $f(x_n) \to f(x_0)$.
But since $x_n \in S$, by definition of $S$, we have $f(x_n) < c$, and therefore

$$f(x_0) = \lim_{n \to \infty} f(x_n) \leq c \checkmark$$

**STEP 3:** Show $f(x_0) \geq c$

First of all, we have $x_0 \neq b$ because $f(x_0) \geq c$ whereas $f(b) > c$, so $x_0$ and $b$ cannot be equal. Therefore $x_0 < b$.

Since $x_0 < b$, for $n$ small enough, we have $t_n =: x_0 + \frac{1}{n} < b$. By definition $t_n \in [a, b]$, $t_n > x_0$ and $t_n \to x_0$. 
Since \( t_n \to x_0 \) and \( f \) is continuous, we have \( f(t_n) \to f(x_0) \).

Moreover, since \( t_n > x_0 \) and \( x_0 = \text{sup}(S) \), we must have \( t_n \notin S \), meaning that (by definition of \( S \)), \( f(t_n) \geq c \).

Therefore, we get

\[
f(x_0) = \lim_{n \to \infty} f(t_n) \geq c\checkmark
\]

Combining STEP 2 and STEP 3, we get \( f(x_0) = c \) \( \square \)

**Note:** The same result holds if you replace \([a, b]\) by any *connected* set. Connected intuitively just means that the set just had one piece. For instance \([a, b]\) is connected but \([0, 1] \cup [2, 3]\) is disconnected; it has two pieces.
4. **Image of an Interval**

**Video:** Image of an interval

Because the Intermediate Value Theorem, it is interesting to figure out what happens when you apply a function to an interval.

**Notation:**

$I$ denotes an interval, such as $I = (0, 1)$ or $[1, 2)$ or $[2, 3]$ or $(3, \infty)$ or even $\mathbb{R}$

**Definition:**

If $I$ is an interval then the **image of $f$ of $I$** (or the range of $f$) is

$$f(I) = \{ f(x) \mid x \in I \}$$
For general $f$, $f(I)$ could be some crazy set (think a fractal or the Cantor set), but it turns out that if $f$ is continuous, then $f(I)$ is very nice:

**Fact:**
If $f$ is continuous, then $f(I)$ is an interval (or a single point)

**Example 1:**
If $f(x) = x^2$ and $I = (-2, 2)$ then

$$f(I) = \{x^2 \mid x \in (-2, 2)\} = [0, 4)$$
Beware: Even though $(-2, 2)$ is open, $f((-2, 2))$ isn’t necessarily open!

**Example 2:**

If $f(x) \equiv 3$ and $I$ is any nonempty interval, then

$$f(I) = \{3\}$$
**Proof:** Let $J = f(I)$ and let $m = \inf(J)$ and $M = \sup(J)$.

Case 1: $m = M$, then $J = \{m\}$ is a single point ✓

Case 2: $m < M$.

**Claim:** $J$ contains the interval $(m, M)$

Then we would be done because we would then have either $J = (m, M)$ or $J = [m, M]$ or $J = (m, M]$ or $J = [m, M]$, depending on whether or not $m = \inf(J)$ and $M = \sup(J)$ are in $J$ or not (here the endpoints
may be infinite).

**Proof of Claim:** Let $c \in (m, M)$, and show $c \in J$.

By assumption $m < c < M$. Since $c > m = \inf(J)$, by definition of inf, there is $y_0 \in J$ such that $y_0 < c$, and since $c < M = \sup(J)$, there is $y_1 \in J$ such that $c < y_1$.

Therefore we get $y_0 < c < y_1$.

Since $y_0 \in J = f(I)$, by definition of $f(I)$, there is $a \in I$ such that $y_0 = f(a)$. Similarly there is $b \in I$ such that $y_1 = f(b)$. 
Since $f$ is continuous and $c$ is between $f(a)$ and $f(b)$, by the Intermediate Value Theorem, there is $x$ between $a \in I$ and $b \in I$ (so $x \in I$ since $I$ is an interval) such that $f(x) = c$, but this means that $c \in f(I) = J$. \hfill \Box