LECTURE 21: SERIES (III)

In this third and final episode of our series-extravaganza, we will prove the two remaining tests for convergence: The Integral Test and the Alternating Series Test.

1. **Integral Test 1**

   **Video:** [Integral Test 1](#)

   This test is *integral* in our understanding of series! It basically says that if an integral is \(\infty\), then the corresponding series is \(\infty\) as well.

   **Integral Test 1:**

   Suppose \(f(x) \geq 0\) is decreasing on \([1, \infty)\), then

   \[
   \int_{1}^{\infty} f(x) \, dx = \infty \Rightarrow \sum_{n=1}^{\infty} f(n) \text{ diverges}
   \]

   *Date:* Friday, May 15, 2020.
Example 1:

Does the $1$-series converge or diverge?

\[ \sum_{n=1}^{\infty} \frac{1}{n} \]

Let $f(x) = \frac{1}{x}$ (so $f(n) = \frac{1}{n}$), then

\[ \int_{1}^{\infty} f(x) \, dx = \int_{1}^{\infty} \frac{1}{x} \, dx = [\ln(x)]_{1}^{\infty} = \ln(\infty) - \ln(1) = \infty - 0 = \infty \]

(We’re being a bit hand-wavy here because we haven’t defined improper integrals, but the result is still the same)

Therefore, by the integral test, $\sum \frac{1}{n}$ diverges.
Proof:

Note: To make things a bit easier to understand, we will do the proof for \( f(x) = \frac{1}{x} \) and show that \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges. The exact same proof works if you simply replace \( \frac{1}{x} \) by \( f(x) \) (see Homework)

Consider the partial sums:

\[
s_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}
\]

Main Idea: Interpret the sum above in terms of areas of rectangles, and compare it with the area under \( f \), namely \( \int_{1}^{\infty} \frac{1}{x} \, dx \).

Start with the rectangle with base \([1, 2]\) and height \( f(1) = 1 \) (left endpoint), which has area \( 1 \times 1 = 1 \).

Then consider the rectangle with base \([2, 3]\) and height \( f(2) = \frac{1}{2} \), which has area \( 1 \times \frac{1}{2} = \frac{1}{2} \)

Continue that way until you have the rectangle with base \([n, n+1]\) and height \( \frac{1}{n} \), which has area \( \frac{1}{n} \)
Then

\[ s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \text{Sum of areas of } n \text{ rectangles} \]

(In the picture, \( s_n \) is the sum of the green and the blue regions)

On the other hand, the sum of the areas is larger than the area under \( f \) from 1 to \( n+1 \) that is \( \int_1^{n+1} f(x) \, dx \). (see the picture above; the area under \( f \) is denoted in green, whereas the areas of the rectangles are in green and blue).

This follows from the fact that \( f \) is decreasing, and therefore on each interval \([k, k+1]\) (with \( k = 1, \ldots, n \)), the left-endpoint is larger than any other value of \( f \) on that interval, and therefore the area of each rectangle is larger than the area under \( f \) on \([k, k+1]\) (and finally sum over \( k \) to get the result).
And therefore

\[ s_n = \sum_{k=1}^{n} \frac{1}{k} \geq \int_{1}^{n+1} f(x)dx =: t_n \]

However

\[ \lim_{n \to \infty} t_n = \lim_{n \to \infty} \int_{1}^{n+1} f(x)dx = \int_{1}^{\infty} f(x)dx = \infty \quad \text{(By assumption)} \]

And therefore, by comparison, \( \lim_{n \to \infty} s_n = \infty \), meaning that \( \sum_{n=1}^{\infty} \frac{1}{n} = \infty \) (by definition of a series)
Corollary:
If $p < 1$, then
\[ \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ diverges} \]

Example 2:
\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} = \infty \]

Proof: Either use the integral test, or notice that if $p < 1$, then, since $n \geq 1$, we have $n^p \leq n$ (Think $\sqrt{n} \leq n$)

Therefore
\[
\sum_{n=1}^{\infty} \frac{1}{n^p} \geq \sum_{n=1}^{\infty} \frac{1}{n}
\]

But since \( \sum \frac{1}{n} = \infty \), we get \( \sum_{n=1}^{\infty} \frac{1}{n^p} = \infty \) by the comparison test.

2. **Integral Test 2**

**Video:** [Integral Test 2](#)

In a similar way, we can use the integral test to show that a series converges:

**Example 3:**

Does the 2–series converge?

\[
\sum_{n=1}^{\infty} \frac{1}{n^2}
\]

Let \( f(x) = \frac{1}{x^2} \), then

\[
\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x^2}dx = \left[ -\frac{1}{x} \right]_1^{\infty} = -\frac{1}{\infty} + \frac{1}{1} = 1 < \infty
\]

Therefore \( \sum \frac{1}{n^2} \) converges.

**Note:** In fact, one can show \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \). Check out this video for a proof if you’re interested: [Sum of \( \frac{1}{n^2} \)](#)
**WARNING:** In general, the value of the integral tells us nothing about the value of the series. For instance, here $\int_1^{\infty} \frac{1}{x^2} \, dx = 1$, but $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, which are not related.

**Integral Test 2:**

Suppose $f(x) \geq 0$ and is decreasing on $[1, \infty)$. Then

$$\int_1^{\infty} f(x) \, dx \text{ converges } \Rightarrow \sum_{n=1}^{\infty} f(n) \text{ converges}$$

**Proof:** This time we’ll illustrate the proof with $\frac{1}{x^2}$.

Consider again the partial sums

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$$

**Recall:**

If the terms of the sequence are $\geq 0$, then

$$(s_n) \text{ converges } \Leftrightarrow (s_n) \text{ is bounded}$$

Hence it is enough to show that $(s_n)$ is bounded.

This time, start with the rectangle with base $[0, 1]$ and height $f(1) = 1$ (right endpoint) which has area 1.

Then consider the rectangle with base $[1, 2]$ and height $f(2) = \frac{1}{4}$.

Continue this way until you get the rectangle with base $[n-1, n]$ and height $f(n) = \frac{1}{n^2}$.
Note: This is not quite the same as before! In the proof before, we took the left endpoints, but here we take the right endpoints.

Therefore:

$$s_n = 1 + \frac{1}{4} + \cdots + \frac{1}{n^2} = \text{Sum of the areas of the rectangles}$$

Note: Since $f$ and $\int_1^\infty f(x)dx$ is only defined on $[1, \infty)$, we need to ignore the first rectangle (which has finite area anyway), so

$$s_n = (\text{Rectangle 1}) + (\text{Rectangles 2 to n}) = 1 + (\text{Rectangles 2 to n})$$

This time, notice that the area under the graph of $f$ from 1 to $n$ is bigger than the sum of the areas of rectangles 2 to $n$ (in the picture above, the areas of the rectangles is in blue, but the area under the graph of $f$ is in blue and green).

This follows again from the fact that $f$ is decreasing: On each rectangle $[k-1, k]$ (with $k = 2, \ldots, n$), $f(k) = \frac{1}{k^2}$ is smaller than any other
value of $f$ in the rectangle, therefore the area under $f$ is bigger than the area of the rectangle (and finally sum over $k = 2, \ldots, n$)

Hence the areas of rectangles 2 to $n$ is $\leq \int_1^n f(x) dx$, and therefore:

$$s_n \leq 1 + \text{Area of Rectangles 2 to } n$$

$$\leq 1 + \int_1^n f(x) dx$$

$$\leq 1 + \int_1^\infty f(x) dx \text{ (since } f \geq 0)$$

Therefore, with $M =: 1 + \int_1^\infty f(x) dx$ we get $0 \leq s_n \leq M$

Hence $|s_n| \leq M$ for all $n$, and so $(s_n)$ is bounded, and therefore $\sum \frac{1}{n^2}$ converges (by the fact above)
Corollary:

If $p > 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges}$$

Proof: This is because

$$\int_{1}^{\infty} \frac{1}{x^p} dx = \int_{1}^{\infty} x^{-p} dx = \left[ \frac{x^{1-p}}{1-p} \right]_1^\infty = 0 - \left( \frac{1}{1-p} \right) = \frac{1}{p-1} < \infty$$

Therefore, by the integral test, we have $\sum \frac{1}{n^p}$ converges.

Combining the two corollaries, we get:

Corollary: [p-series]

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges } \iff p > 1$$

3. Alternating Series Test

Video: [Alternating Series Test](#)

Finally, there is a wonderful test called the alternating series test, which basically says that all alternating series converge.
**Definition:**

An alternating series is a series of the form \( \sum (-1)^n a_n \) or \( \sum (-1)^{n+1} a_n \), where \( a_n \geq 0 \).

**Example 4:**

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{1}{n} \right) = 1 - \frac{1}{2} + \frac{1}{3} - \ldots
\]

is an alternating series with \( a_n = \frac{1}{n} \).

In other words, an alternating series alternates between positive and negative values.

**Alternating Series Test:**

If \( a_n \geq 0 \), is decreasing, and \( a_n \geq 0 \) then \( \sum (-1)^n a_n \) converges.

**Example 5:**

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}
\]

converges because \( a_n = \frac{1}{n} \geq 0 \) and is decreasing.

**Proof:** For this, we need to use the Cauchy Criterion from two lectures ago:

**Recall: Cauchy Criterion**

\( \sum a_n \) converges if and only if for all \( \epsilon > 0 \) there is \( N \) such that if \( n \geq m > N \), then \( |\sum_{k=m}^{n} a_n| < \epsilon \).
Let \( \epsilon > 0 \) be given. Since \( a_n \to 0 \) there some \( N \) such that \( a_N < \epsilon \).

With \( N \) as above, if \( n \geq m > N \), let’s show the following:

**Claim:**

\[
\left| \sum_{k=m}^{n} (-1)^k a_k \right| \leq a_N
\]

Then we would be done because we’d have

\[
\left| \sum_{k=m}^{n} (-1)^k a_k \right| \leq a_N < \epsilon
\]

Therefore \( \sum (-1)^n a_n \) converges by the Cauchy criterion.

**Proof of Claim:** Notice:
\[ \left| \sum_{k=m}^{n} (-1)^k a_k \right| = \left| (-1)^m a_m + (-1)^{m+1} a_{m+1} + \cdots + (-1)^n a_n \right| \]

\[ = \left| (-1)^m \left( a_m - a_{m+1} + \cdots + (-1)^{n-m} a_n \right) \pm 1 \right| \]

\[ = \left| a_m - a_{m+1} + \cdots (-1)^{n-m} a_n \right| \]

**Case 1:** \( n - m \) is odd

Then \( (-1)^{n-m} = -1 \) (so our sum looks something like \( a_3 - a_4 + a_5 - a_6 \)),
and so

\[ \left| a_m - a_{m+1} + \cdots (-1)^{n-m} a_n \right| = \left| a_m - a_{m+1} + \cdots - a_n \right| \]

\[ = \left| \underbrace{a_m - a_{m+1}}_{\geq 0} + \cdots + \underbrace{(a_{n-1} - a_n)}_{\geq 0} \right| \]

\[ = a_m - a_{m+1} + \cdots + a_{n-1} - a_n \]

(Here we used the fact that \( (a_n) \) is decreasing, and so, for example, \( a_m \geq a_{m+1} \), hence \( a_m - a_{m+1} \geq 0 \))
On the other hand, we can write this as (Analogy: Think \( a_3 - a_4 + a_5 - a_6 = a_3 - (a_4 - a_5) - a_6 \))

\[
a_m - a_{m+1} + \cdots - a_n = a_m - a_{m+1} + a_{m+2} + \cdots - a_{n-2} + a_{n-1} - a_n
\]

\[
= a_m - \left( a_{m+1} - a_{m+2} \right) - \cdots - \left( a_{n-2} - a_{n-1} \right) - a_n
\]

\[
\leq a_m - a_n
\]

\[
\geq 0
\]

\[
\leq a_m
\]

\[
\leq a_N \checkmark
\]

(the last inequality follows because \( m > N \) and \((a_n)\) is decreasing) Hence

\[
\left| \sum_{k=m}^{n} (-1)^k a_k \right| = |a_m - a_{m+1} + \cdots - a_n| \leq a_N \checkmark
\]
And we are done in the case where \( n - m \) is odd

**Case 2: \( n - m \) is even**

Then \((-1)^{n-m} = 1\) (so our sum looks something like \( a_3 - a_4 + a_5 - a_6 + a_7 = (a_3 - a_4) + (a_5 - a_6) + a_7 \))

Therefore:

\[
|a_m - a_{m+1} + \ldots (-1)^{n-m}a_n| = |a_m - a_{m+1} + \cdots + a_n| \\
= \left| \left( a_m - a_{m+1} \right) + \cdots + \left( a_{n-2} - a_{n-1} \right) + a_n \right| \\
= a_m - a_{m+1} + \cdots + a_n
\]

On the other hand, we can write this as (Analogy: Think \( a_3 - a_4 + a_5 - a_6 + a_7 = a_3 - (a_4 - a_5) - (a_6 - a_7) \))

\[
a_m - a_{m+1} + \cdots + a_n = a_m - a_{m+1} + a_{m+2} + \cdots - a_{n-1} + a_n \\
= a_m - \left( a_{m+1} - a_{m+2} \right) - \cdots - \left( a_{n-1} - a_n \right) \\
\leq a_m \\
\leq a_N
\]

Hence

\[
\left| \sum_{k=m}^{n} (-1)^k a_k \right| = |a_m - a_{m+1} + \cdots + a_n| \leq a_N \checkmark
\]

And we are done in this case as well \( \square \)
4. Error Estimate

In fact, more can be said about the convergence of an alternating series:

**Error Estimate:**

Let \( \sum (-1)^n a_n \) be an alternating series that converges to \( S \), then

\[
|s_n - S| \leq a_{n+1}
\]

(Where \( s_n \) is the \( n \)-th partial sum)

That is the difference/error between the sum of the first \( n \) terms and \( S \) is at most equal to the next term \( a_{n+1} \)

**Example 6:**

One can show (using power series, see Math 140B) that

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} + \cdots = \ln(2)
\]

What the above error estimate (with \( n = 4 \)) is saying is that

\[
|s_4 - S| = \left| \left( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \right) - \ln(2) \right| \leq a_5 = \frac{1}{5}
\]

So with the first 4 terms, we are at most \( \frac{1}{5} = 0.2 \) away from the correct answer.

**Intuitively:** This makes sense because the partial sums \( s_n \) jump back and forth between values bigger than \( S \) and smaller than \( S \) (since the series is alternating).
Hence $|s_n - S|$ should be smaller than $|s_{n+1} - s_n| = a_{n+1}$ (since $s_{n+1} = s_n + (-1)^{n+1}a_{n+1}$, since the sum of the first $n + 1$ terms is the sum of the first $n$ terms plus the last term)

**Proof of Error Estimate:** (optional; not in the video)

Notice that in the proof above, we have shown that for all $m$ and all $n \geq m$

$$\left| \sum_{k=m}^{n} (-1)^k a_k \right| \leq a_m$$

(We only used $a_N$ in the very last step)

Therefore, letting $n \to \infty$ in the above, we get

$$\left| \sum_{k=m}^{\infty} (-1)^k a_k \right| \leq a_m$$

But since the sum from $m$ to $\infty$ is the same thing as the sum from 1 to $\infty$ minus the sum from 1 to $m - 1$, we get:
∞ ∑ \[ k=m \] \((−1)^k a_k\) = \(\left(\sum_{k=1}^{\infty} (-1)^k a_k\right) - \left(\sum_{k=1}^{m-1} (-1)^k a_k\right) = S - s_{m−1}\)

Where \(S\) is the value of the series, and \(s_{m−1}\) is the partial sum. Therefore, we get

\[ |S - s_{m−1}| \leq a_m \]

And so, writing \(n\) instead of \(m−1\) (and \(n+1\) instead of \(m\)) we therefore obtain

\[ |s_n - S| \leq a_{n+1} \quad \Box \]

5. **Conditional Convergence**

The alternating series test is useful to show that a series converges conditionally:
Recall:

(1) \( \sum a_n \) is **absolutely convergent** if \( \sum |a_n| \) is convergent

(2) Absolute Convergence \( \Rightarrow \) Convergent

**Conditionally Convergent**

Series that are convergent, but not absolutely convergent are called *conditionally convergent*

**Definition:**

\[ \sum a_n \text{ is conditionally convergent if } \sum a_n \text{ converges, but } \sum |a_n| \text{ diverges} \]

**Example 7:**

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} + \ldots \]
This series is convergent by the alternating series test (see above), but

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \]

is divergent, and therefore \( \sum \frac{(-1)^{n+1}}{n} \) is conditionally convergent.

Optional Fun Fact:
If \( \sum a_n \) is conditionally convergent, then you can rearrange \( \sum a_n \) to get any limit you want!

For example, a rearrangement of

\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots \]

Is

\[ 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots \]

Not only can it be shown that the two series here have two different limits, but given any \( a \in \mathbb{R} \) (including \( \pm \infty \)), there is a rearrangement of the first series that converges to \( a \) (WOW)

Congratulations, we are now officially done with Chapter 2! Next time we’ll start Chapter 3, the magical world of Continuous Functions.