LECTURE 18: METRIC SPACES (III)

In this last exciting episode of the Metric Space trilogy, we’ll discuss a really abstract, but super powerful concept: Compactness.

1. The Cantor Set

Video: [The Cantor Set]

Optional: [Full Version]

But before we do that, let me quickly introduce you to the single most important set in Analysis: The Cantor Set.

This set is constructed in stages.

STEP 1: Start with $F_1 = [0, 1]$

\[
0 \quad \quad \quad \quad \quad \quad 1 \quad \quad \quad \quad \quad \quad F_1
\]

STEP 2: Remove the middle third $(1/3, 2/3)$ of $F_1$ to get two pieces $F_2 = [0, 1/3] \cup [2/3, 1]$

Date: Friday, May 8, 2020.
STEP 3: Remove the middle of each piece of $F_2$ to get 4 pieces $F_3 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$

STEP 4: Continue so on and so forth: Given $F_n$, define $F_{n+1}$ by removing the middle third of each sub-interval of $F_n$
And in this way we get a decreasing sequence of intervals $F_1 \supseteq F_2 \supseteq \ldots$

Definition:
The Cantor set is:

$$F = \bigcap_{n=1}^{\infty} F_n$$
## Neat Facts:

1. $F$ is closed
2. $F$ is nonempty
3. $F$ has size 0
4. $F^\circ = \emptyset$
5. $F$ is compact (see second half of lecture)
6. $F$ is uncountable

### Proof:

1. $F$ is closed because it is the intersection of closed set

2. $F$ is nonempty by the finite intersection property (see last time)

3. (This was on your HW) Since each $F_n$ consists of $2^{n-1}$ pieces of length $(\frac{1}{3})^{n-1}$, each $F_n$ has $2^{n-1} \times (\frac{1}{3})^{n-1} = (\frac{2}{3})^{n-1}$ which goes to 0

4. (This was on your HW) Suppose $x \in F^\circ$, then there is $r > 0$ such that $B(x, r) = (x - r, x + r) \subseteq F$. But since $(x - r, x + r)$ has size $x + r - (x - r) = 2r > 0$, $F$ must also have size at least $2r$, which contradicts the fact that $F$ has size 0.

5. This is because $F$ is closed and bounded, so by the Heine-Borel Theorem (see the second half of the lecture), $F$ is compact

6. $F$ is uncountable (will be explained below)

### Aside: Also $F$ is perfect and totally disconnected (whatever those terms mean). Moreover, it turns out that any metric space can be
thought of as a subset of the Cantor set.

Optional Discussion:

More interestingly, there’s a very natural characterization between the Cantor set and ternary expansions (= expansions in base 3):

Definitions:

**Decimal expansion:** An expression like 0.1248... where you use digits from 0 to 9

**Binary expansion:** An expression like 0.11001... where you use only digits 0 and 1, used a lot in computer science with bits.

**Ternary expansion:** An expression like 0.120211020... where you use only digits 0, 1, and 2

The ternary expansion of $\frac{1}{3}$ is 0.1 and the ternary expansion of $\frac{2}{3}$ is 0.2. So by removing the middle term in $F_1$ to get $F_2$, we’re essentially throwing away all the numbers of the form 0.1***, so whose first digit is 1

\[^1\text{Technically 0.1 is in } F_2, \text{ but you can just write that as 0.022222...} \]
Similarly, in $F_2$ we’re throwing away numbers of the form $0.01***$ and $0.21***$, so whose second digit is 1.
Cool Facts:

$F$ is just the set of all numbers between 0 and 1 that have no 1 in their expansion.

For instance, 0.022022 is in $F$, but 0.02210202 is not in it.

What makes this super neat is that the function $f(x) = \frac{x}{2}$ turns elements in $F$ into numbers with binary expansions; for instance $f(0.22022) = 0.11101$.

Since every element of $[0,1]$ has a binary expansion, $f$ is actually a bijection (= one-to-one correspondence) between $F$ and $[0,1]$.

Therefore $F$ has the same cardinality as $[0,1]$, and is hence uncountable!
2. COMPACTNESS

Video: Compactness

Warning: This section is quite difficult! But it is very important and you need to know it for the exams.

As a grand finale of our metric space extravaganza, let’s discuss one of the most powerful concepts of Analysis: Compactness. This notion has to do with open covers, which we define now.

Intuitively: An open cover of $E$ is just a family of open sets which “covers” $E$, as in the following picture (here $E$ is the oval disk, $U$ is a colored disk, and $\mathcal{U}$ is the collection of all the disks)
Definition:
Let $\mathcal{U}$ be a family of open sets $U$.

Then $\mathcal{U}$ is an **open** cover for $E$ if

$$E \subseteq \bigcup \{U \mid U \in \mathcal{U}\}$$

That is: If $x \in E$, then there is some $U \in \mathcal{U}$ with $x \in U$.

In other words, $E$ is included in the union of all the $U$'s.

**Interpretation:** Think of the $U$’s as patches that cover the region $E$.

**Example 1:** Let $E = \mathbb{R}$, then the following is an open cover for $E$:

$$\mathcal{U} = \{\ldots, (-3, -1), (-2, 0), (-1, 1), (0, 2), (1, 3), \ldots\}$$

$$= \{(m - 1, m + 1) \mid m \in \mathbb{Z}\}$$
Example 2:

\[ E = \mathbb{R}^k \]

\[ \mathcal{U} = \{ B((m, n), 1) \mid m, n \in \mathbb{Z} \} \]

That is, the family of open balls centered at pairs of integers (like (2, 2)) and radius 1

Some covers are better than others! For instance, imagine you’re paying $1,000 per patch. Then ideally you’d like to cover the same space, but with as few patches as possible. The idea behind this is expressed in the notion of a sub-cover:

**Definition:**

\( \mathcal{V} \) is a **subcover** of \( \mathcal{U} \) if \( \mathcal{V} \) is a subset of \( \mathcal{U} \) that also covers \( E \)
In the picture above, all the sets are in $\mathcal{U}$, but only the sets in red, blue, yellow, green, and in orange are in $\mathcal{V}$. So $\mathcal{V}$ is a subset of $\mathcal{U}$. Moreover, since $\mathcal{V}$ also covers $E$, $\mathcal{V}$ is a subcover of $\mathcal{U}$.

In some sense, $\mathcal{V}$ is better than $\mathcal{U}$. $\mathcal{V}$ still does the job of covering $E$, but with fewer elements.

**Example:**

Let $E = [0, 1]$ and consider

$$\mathcal{U} = \{(-1, 1), (0, 2), (1, 3)\}$$
Then the following is a subcover of $\mathcal{U}$

$$\mathcal{V} = \{(-1, 1), (0, 2)\}$$

**Note:** In some sense, the $(1, 3)$ one is redundant; you wouldn’t really pay $1,000 for it because $(-1, 1)$ and $(0, 2)$ are enough to cover $E$.

**Definition:**

A **finite** subcover $\mathcal{V}$ is a subcover with finitely many elements

**Example:** In the above example,

$$\mathcal{V} = \{(-1, 1), (0, 2)\}$$

is a finite subcover because it only has 2 elements/patches: $(-1, 1)$ and $(0, 2)$

But

$$\mathcal{V} = \{\ldots, (-2, 0), (-1, 1), (0, 2), (1, 3), (2, 4), \ldots \}$$
would not be a finite subcover because it has infinitely many elements (and it’s not a subset of $U$ anyway)

We are now finally ready for the definition of compactness:

**Definition:**

A set $E$ is **compact** if every open cover $U$ of $E$ has a finite subcover $V$.

(In the above picture, all the green and red patches belong to $U$, but only the red ones belong to $V$)

In other words, compact sets are efficient/cost-effective: We *never* need infinitely many patches to cover $E$. Finitely many sets will *always* do the job.
**Note:** In practice, it’s hard to show that a set is compact (but see below), because you’d have to show that *every* cover has a finite sub-cover. It is much easier to show that sets are not compact:

Non-Example 1:

$E = \mathbb{R}$ is not compact.

We must find *one* cover that does not have a finite sub-cover.

Consider the following cover $U$ of $\mathbb{R}$:

$U = \{(-1, 1), (-2, 2), \ldots \} = \{(-n, n) \mid n \in \mathbb{R}\}$

Suppose $U$ had a finite sub-cover

$\mathcal{V} = \{(-n_1, n_1), \ldots, (-n_k, n_k)\}$

Let $N = \max \{n_1, \ldots, n_k\}$

Then union of $\mathcal{V}$ is $(-N, N)$. 
But since $\mathcal{V}$ is a sub-cover, $\mathbb{R}$ must be contained in that union, so $\mathbb{R} \subseteq (-N, N)$, which makes no sense because $N + 1 \in \mathbb{R}$ but $N + 1 \notin (-N, N)$ must contain $\mathbb{R}$, which isn’t possible $\Rightarrow \Leftarrow$.

Hence $\mathcal{U}$ has no finite sub-cover, and therefore $E = \mathbb{R}$ is not compact.

**Non-Example 2:**

$E = (0, 1)$ is not compact.

Consider the following cover $\mathcal{U}$ of $(0, 1)$:

$$\mathcal{U} = \left\{ \left( \frac{1}{n}, 1 \right) \mid n \geq 2 \right\} = \left\{ \left( \frac{1}{2}, 1 \right), \left( \frac{1}{3}, 1 \right), \left( \frac{1}{4}, 1 \right), \ldots \right\}$$
Suppose $\mathcal{U}$ has a finite sub-cover $\mathcal{V}$, where

$$\mathcal{V} = \left\{ \left( \frac{1}{n_1}, 1 \right), \ldots, \left( \frac{1}{n_k}, 1 \right) \right\}$$

Let $N = \max \{n_1, \ldots, n_k\} > 0$

Then the union of all the sets in $\mathcal{V}$ is $\left( \frac{1}{N}, 1 \right)$, but this cannot contain $(0, 1)$ since $x = \frac{1}{N+1}$ is in $(0, 1)$ but not in $\left( \frac{1}{N}, 1 \right)$, which contradicts $\mathcal{V}$ being a subcover of $\mathcal{U}$.
But then what is a compact set? This is hard to answer, but luckily, at least in the case of $\mathbb{R}^k$, there’s a wonderful theorem called the Heine-Borel Theorem (see below) that will take care of that.

**Example 3:**

The following sets are compact in $\mathbb{R}^k$:

1. Closed intervals $[a, b]$
2. Boxes like $[1, 2] \times [3, 4]$
3. Closed balls $B(x, r)$ in $\mathbb{R}^k$

### 3. Properties of Compactness

**Video:** [Compactness Properties](#)

In this section, we show that compact sets must be closed and bounded.

**Definition**

A (nonempty) set $E$ is **bounded** if there is $x \in E$ and $r > 0$ such that $E \subseteq B(x, r)$. 
That is, $E$ is included in some large ball.\(^2\)

**Non-Example:**

$\mathbb{R}^k$ is not bounded; there’s no way to fit all of $\mathbb{R}^k$ inside a ball.

**Fact 1:**

If $E$ is compact, then $E$ is bounded

**Note:** This is yet another reason why $\mathbb{R}$ (or $\mathbb{R}^k$) are not compact!

**Proof:** Suppose $E$ is compact, and let\(^3\) $x \in E$.

Consider the following cover $\mathcal{U}$ of $E$:

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\(^2\)This looks slightly different from the definition in the textbook, but it’s actually equivalent. The book’s definition is that $E$ is contained in a large box, and mine says that $E$ is contained in a large ball.

\(^3\) $\emptyset$ is compact, so suppose $E \neq \emptyset$. 

\[ \mathcal{U} = \{B(x, n) \mid n \in \mathbb{N}\} \]

Since \( E \) is compact, \( \mathcal{U} \) has a finite sub-cover:

\[ \mathcal{V} = \{B(x, n_1), \ldots, B(x, n_k)\} \]

Let \( N = \max\{n_1, \ldots, n_k\} > 0 \)
Then, on the one hand, the union of $\mathcal{V}$ is $B(x, N)$.

On the other hand, because $\mathcal{V}$ is a sub-cover, that union must contain $E$, and therefore $E \subseteq B(x, N)$.

Hence with $r = N$, we found $r$ such that $E \subseteq B(x, r)$ and hence $E$ is bounded $\square$

**Fact 2:**

If $E$ is compact, then $E$ is closed

**Note:** This is yet another reason why $(0, 1)$ is not compact; it is not closed.

**Proof:** To show $E$ is closed, it is enough to show that $E^c$ is open.
Suppose \( x \in E^c \). In order to show \( E^c \) is open, we need to find \( r \) such that \( B(x, r) \subseteq E^c \).

Consider the following cover of \( E \):

\[
U = \{ U_n \mid n \in \mathbb{N} \}
\]

Where

\[
U_n = \left\{ y \mid d(y, x) > \frac{1}{n} \right\}
\]

\( U_n \) consists of all the points that are \textit{at least} \( \frac{1}{n} \) away from \( x \), think of it as \( x \) being repulsive/repellant; no one wants to get close to \( x \).
Then, since each $U_n$ is the complement of $\{ y \mid d(y, x) \leq \frac{1}{n} \}$ (which is closed), each $U_n$ is open.

Moreover the union of the $U_n$ is $\{x\}^c$, which therefore covers $E$ since $x \notin E$. 

\[ n = 1 \quad U_1 \quad n = 2 \quad U_2 \quad n = 3 \quad U_3 \]
Since \( E \) is compact, \( \mathcal{U} \) has a finite sub-cover

\[
\mathcal{V} = \{ U_{n_1}, \ldots, U_{n_k} \}
\]

Let \( N = \max \{ n_1, \ldots, n_k \} > 0 \). Then the union of \( \mathcal{V} \) is

\[
U_N = \left\{ y \mid d(y, x) > \frac{1}{N} \right\}
\]

But since \( \mathcal{V} \) covers \( E \), we have that \( E \) is included in that union, that is \( E \subseteq U_N \) and so \( U_N^c \subseteq E^c \) (see picture above).
But by definition of $U_N$, we have:

$$U_N^c = \left\{ y \mid d(y, x) \leq \frac{1}{N} \right\}$$

(The opposite of “being at least $\frac{1}{N}$ away from $x$” is “being at most $\frac{1}{N}$ away from $x$”)

Therefore

$$U_N^c \subseteq E^c$$

But then notice that

$$B \left( x, \frac{1}{N} \right) = \{ y \mid d(y, x) < r \} \subseteq \{ y \mid d(y, x) \leq r \} \subseteq E^c$$

Hence, if we let $r = \frac{1}{N}$, we get $B(x, r) \subseteq E^c$, which is exactly what we wanted to show! (we needed to find some ball centered at $x$ that is included in $E^c$)

Hence $E^c$ is open, so $E$ is closed

**WARNING:** For a general metric space, if $E$ is closed and bounded, then $E$ is not necessarily compact! (see Homework). However, it turns out that this is true in $\mathbb{R}^k$:

4. **The Heine-Borel Theorem**

**Video:** [Heine-Borel Theorem](#)
Note: The proof below is optional, but really interesting, and should give you a nice taste of what analysis is like. That said, definitely know the statement of the theorem.

From the above, we know that if a set is compact, then it is closed and bounded. The Heine-Borel Theorem says that in $\mathbb{R}^k$, the two are actually equivalent!

**Heine-Borel Theorem:**

A subset $E$ of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.

**Examples:**

The following subsets are compact, since they are closed and bounded:

1. Closed intervals $[a, b]$ in $\mathbb{R}$
2. Boxes like $[1, 2] \times [3, 4]$ (see below)
3. Closed balls $B(x, r)$ in $\mathbb{R}^k$
4. Spheres in $\mathbb{R}^k$

**Proof:** (optional)

$(\Rightarrow)$ Done above

$(\Leftarrow)$ Suppose $E$ is closed and bounded.

First of all, since $E$ is bounded, there is $x \in E$ and $r > 0$ large such that $E \subseteq B(x, r)$. 
Moreover, every ball can be put in a box $F = [a_1, b_1] \times [a_2, b_2] \times [a_m, b_m]$ for some $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$.

Therefore, we get $E \subseteq B(x, r) \subseteq F$, so $E \subseteq F$, where $F$ is a box.

**Fact:**

A closed subset of a compact set is compact.

Therefore, if we show that $F$ is compact, then since $E$ is closed, it follows that $E$ is compact, and we would be done.

To show this, we’ll show the following general result:

**Fact:**

Boxes are compact.

Before we show this, we need a couple of general facts about boxes:

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4You can check $a_i = x_i - r$ and $b_i = x_i + r$ works, where $x = (x_1, \ldots, x_n)$
Definition:
The diameter of $F$ is

$$\delta = d((a_1, \ldots, a_k), (b_1, \ldots, b_k)) = \sqrt{(b_1 - a_1)^2 + \cdots + (b_k - a_k)^2}$$

Note that $\delta$ is the largest possible distance between points in $F$, and therefore:

Remark:
For any $x \in F$, $F \subseteq B(x, \delta)$
Theorem: Boxes are compact

Proof: Let $F = [a_1, b_1] \times \ldots \times [a_k, b_k]$ be a box in $\mathbb{R}^k$

STEP 1: Suppose, for sake of contradiction, that $F$ is not compact

Then there is an open cover $\mathcal{U}$ of $F$ that has no finite sub-cover.

Notice that we can split $F$ into $2^k$ sub-boxes of diameter $\frac{\delta}{2}$ as follows. The picture below illustrates the case $k = 2$, in which we get $2^2 = 4$ sub-boxes (= quadrants), but in the case $k = 8$ we get $2^3 = 8$ octants, etc.
If all the sub-boxes of $F$ had a finite sub-cover, then by taking the union of the $2^k$ finite subcovers, you would get that $F$ has have a finite sub-cover $\Rightarrow \Leftarrow$. 

4 finite sub-covers = Finite sub-cover for $F$
Hence one of the sub-boxes, let’s call it \( F_1 \) cannot be have a finite sub-cover.

\[ \text{No finite sub-cover} \]

Note that \( F_1 \subseteq F \)

**STEP 2:** Now since \( F_1 \) is a box of diameter \( \frac{\delta}{2} \), \( F_1 \) is the union of \( 2^k \) sub-boxes of diameter \( \frac{\delta}{4} \).

Repeating the argument above, we get there is a box \( F_2 \) of diameter \( \frac{\delta}{4} \) that doesn’t have a finite sub-cover \( \mathcal{U} \). Note that \( F_2 \subseteq F_1 \).
**STEP 3:** Continuing in this fashion, we obtain a decreasing sequence $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ of boxes such that each $F_n$ has diameter $\frac{\delta}{2^n}$ and that doesn’t have a finite sub-cover.

But since each $F_n$ is closed (boxes are closed), nonempty, and bounded, by the Finite Intersection Property $\bigcap_{n=1}^{\infty} F_n$ is nonempty, so there is $x \in \bigcap_{n=1}^{\infty} F_n$. 
**STEP 4:** Since \( \mathcal{U} \) covers \( F \) and \( x \in F \), there must be \( U \in \mathcal{U} \) such that \( x \in U \).

But since \( U \) is open, there is \( r > 0 \) such that \( B(x, r) \subseteq U \).

Now let \( n \) be large enough so that \( \frac{\delta}{2^n} < r \).
Then since \( x \in F_n \) (by definition of intersection) and the diameter of \( F_n \) is \( \frac{\delta}{2^n} \), by the remark in the green box preceding the proof, \( F_n \subseteq B(x, \frac{\delta}{2^n}) \) (see the remark preceding the proof), and therefore, since \( r > \frac{\delta}{2^n} \), we have

\[
F_n \subseteq B(x, \frac{\delta}{2^n}) \subseteq B(x, r) \subseteq U
\]

Hence \( F_n \subseteq U \).

But then this means that the one-element set \( \mathcal{V} = \{U\} \) covers \( F_n \). But this contradicts the fact that, by construction, \( F_n \) doesn’t have a finite sub-cover (Remember that we chose \( F_n \) so that there is no finite sub-cover covering \( F_n \) !) \( \Rightarrow \Leftarrow \)

This contradicts the assumption that \( \mathcal{U} \) does not have a finite subcover, and therefore \( \mathcal{U} \) has a finite subcover, and so \( F \) is compact \( \square \)

Congratulations, with this we are officially done with our metric space adventure!