MATH 140A – HOMEWORK 9

Reading: Section 17. This homework set only covers one section, in order to give you a more relaxing Memorial day 😊

- **Section 17**: 8 (see Note), 9, 10(a)(b) (see Note), 12, 13, 14, AP1, AP2, AP3, AP4 (Optional: AP5, AP6, AP7, AP8, AP9, AP10)

Notes: For Problem 8(b), please do this part directly, without using (a) (use the definition of min)

Note: For Problem 10(a), please do this using both the sequence definition and the $\epsilon - \delta$ definition.

**Additional Problem 1**: Use the $\epsilon - \delta$ definition of continuity to prove that

(a) $f(x) = |x|$ is continuous

(b) $f(x) = \frac{1}{x}$ is continuous at $x_0$, for all $x_0 \neq 0$

(c) $f(x) = \sqrt{x}$ is continuous at $x_0$, for all $x_0 > 0$

**Definition:**

$f : \mathbb{R} \to \mathbb{R}$ is **Lipschitz** if there is a constant $C > 0$ such that for all $a$ and $b$, we have

$$|f(b) - f(a)| \leq C |b - a|$$

*Date: Due: Thursday, May 28, 2020.*
Additional Problem 2: Show that if $f$ is Lipschitz then $f$ is continuous.

**Definition:**

If $f : \mathbb{R} \to \mathbb{R}$, and $U$ is any subset of $\mathbb{R}$, then the **pre-image** $f^{-1}(U)$ is defined by

\[ x \in f^{-1}(U) \iff f(x) \in U \]

**Note:** The above definition works for *any* function $f$, not just invertible ones!

**Example:** $f(x) = 2x + 3$, then $f^{-1}((5, 9)) = (1, 3)$ because

\[
x \in f^{-1}((5, 9)) \iff f(x) \in (5, 9) \\
\iff 5 < 2x + 3 < 9 \\
\iff 2 < 2x < 6 \\
\iff 1 < x < 3
\]

Additional Problem 3: Calculate $f^{-1}(U)$ for the following functions $f$ and the following sets $U$

(a) $f(x) = 3x + 7$, $U = (7, 10)$

(b) $f(x) = x^2$, $U = (-1, 4)$

(c) $f(x) = \sin(x)$, $U = (0, 1)$

**Note:** Observe that in all of the examples, both $U$ and $f^{-1}(U)$ are open! This is precisely because $f$ is continuous (in topology, this is taken as the *definition* of continuity, since it only involves open sets).
Fact: (do not prove)

\[ f : \mathbb{R} \to \mathbb{R} \text{ is continuous if and only if} \]
\[ U \text{ is open } \Rightarrow f^{-1}(U) \text{ is open} \]

Additional Problem 4: To illustrate the elegance of the above definition, let’s give a quick proof of the fact that composition of continuous functions are continuous

(a) If \( f \) and \( g \) are any functions (not necessarily invertible), prove that

\[ (g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \]

(b) Use (a) and the definition above to show that if \( f \) and \( g \) are continuous, then \( g \circ f \) is continuous

Optional Additional Problem 5: Prove the above fact (before AP4) about continuous functions

Optional Additional Problem 6: Prove that, for any function \( f \) and any sets \( A \) and \( B \), we have

(a) \( f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \)

(b) \( f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \)

(c) \( f^{-1}(A^c) = (f^{-1}(A))^c \)

Definition:

Given a function \( f \) and a subset \( A \) of \( \mathbb{R} \), we define

\[ f(A) = \{ f(x) \mid x \in A \} \]
Optional Additional Problem 7: Here’s a nice exercise using compactness and pre-images

(a) Show that if $K$ is (covering) compact and $f$ is continuous, then $f(K)$ is (compact)

(b) Is there a continuous function $f$ with domain $[0, 1]$ and range $(0, 1)$?

(c) Show that any continuous function from $[a, b]$ to $\mathbb{R}$ must be bounded

The definition of continuity can be generalized to metric spaces

**Definition:**

If $(S, d)$ and $(S', d')$ are metric spaces with $f : S \to S'$

Then $f$ is **continuous** at $x_0 \in S$ if for all $\epsilon > 0$ there is $\delta > 0$ such that for all $x$,

$$d(x, x_0) < \delta \Rightarrow d'(f(x), f(x_0)) < \epsilon$$

$f$ is **continuous** if $f$ is continuous at $x_0$ for all $x_0 \in S$

Optional Additional Problem 8: Let $(S, d)$ be any metric space, and consider $(\mathbb{R}^k, d')$ where $d'$ is the usual metric:

$$d'((x_1, \ldots, x_k), (y_1, \ldots, y_k)) = \sqrt{\sum_{j=1}^{k} (y_j - x_j)^2}$$

Show that $f = (f_1, \ldots, f_k) : S \to \mathbb{R}^k$ is continuous if and only if each component $f_j : S \to \mathbb{R}$ is continuous (where $\mathbb{R}$ is equipped with the
usual metric).

Optional Additional Problem 9: Let \((S, d)\) be \(\mathbb{R}\) equipped with the discrete metric

\[
d(x, y) = \begin{cases} 
1 & \text{if } x = y \\
0 & \text{if } x \neq y
\end{cases}
\]

And let \((S', d')\) be any metric space. Show that any function \(f : S \to S'\) must be continuous.

Optional Additional Problem 10: This problem is taken from the Berkeley Pre-lim, which is an exam given to first year graduate students at Berkeley, and is therefore quite challenging.

Suppose that \(f : \mathbb{R}^k \to \mathbb{R}\) (with their usual metrics) satisfies the following two conditions:

1. For each compact set \(K\), \(f(K)\) is compact.
2. For any nested decreasing sequence of compact sets \(K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots\), we have

\[
f \left( \bigcap K_n \right) = \bigcap f(K_n)
\]

Show that \(f\) is continuous.
Hints:

17.8(b) Do it by cases, first assuming $f(x) \leq g(x)$ and then assuming $g(x) \leq f(x)$. Remember that I did the version with max in the following video: Max is continuous

17.9(a)(d) Do the usual trick of assuming $|x - x_0| < 1$ and therefore

$$|x| = |x - x_0 + x_0| \leq |x - x_0| + |x_0| = 1 + |x_0|$$

Yes, the constant gets unusually big for (d)

17.10(a) I did a very similar problem in the following video: Not continuous

17.12 For (a), remember that $\mathbb{Q}$ is dense in $\mathbb{R}$, and for (b), consider $f(x) - g(x)$ and use the result in (a)

17.13 For (a), if $x_0$ is irrational, use $\mathbb{Q}$ is dense in $\mathbb{R}$, or, if $x_0$ is rational use that $x_n = x_0 + \frac{\sqrt{2}}{n}$ is a sequence of irrational numbers converging to $x_0$. For (b), use a similar argument, except for $x_0 = 0$ where you use $|h(x)| \leq |x|$

17.14 I’m surprised the book didn’t put any hints here, because it’s a nontrivial problem! First of all, if $x_0$ is rational use the sequence $x_n = x_0 + \frac{\sqrt{2}}{n}$.

To show $f$ is continuous at irrational points $x_0$, use an $\epsilon - \delta$ argument as follows:

Let $\epsilon > 0$ be given, and let $N$ be such that $\frac{1}{N} < \epsilon$. Then, choose $\delta$ so small that there are no integers in $(x_0 - \delta, x_0 + \delta)$, and now choose $\delta$ so small that there are no fractions with denominator 2 in $(x_0 - \delta, x_0 + \delta)$,
and choose \( \delta \) even smaller that there are no fractions with denominator 3 in \((x_0 - \delta, x_0 + \delta)\), and so on, until there are no fractions with denominator \( N \) in \((x_0 - \delta, x_0 + \delta)\).

If \( |x - x_0| < \delta \) and \( x = \frac{p}{q} \) is rational, show \( q \geq N + 1 \) and conclude. And what if \( x \) is irrational?

**AP 1:** For (b), this is similar to the part in the lecture where I showed that \( \frac{1}{f} \) is continuous: you have to assume \( |x - x_0| < \frac{|x_0|}{2} \) and solve for \( |x| \) using the reverse triangle inequality. For (d), multiply \( \sqrt{x} - \sqrt{x_0} \) by \( \frac{x + x_0}{\sqrt{x} + \sqrt{x_0}} \).

**AP 4(a):** If \( x \in (g \circ f)^{-1}(U) \), then \((g \circ f)(x) \in U\), so \( g(f(x)) \in U \), but then what can you tell me about \( f(x) \)? and then what can you tell me about \( x \)? Likewise, if \( x \in f^{-1}(g^{-1}(U)) \), what can you tell me about \( f(x) \)? What can you tell me about \( g(f(x)) \)? So what can you tell me about \( x \)?

**AP 5:**

(⇒) Suppose \( f \) is continuous and \( U \) is open, and let’s show \( f^{-1}(U) \) is open. For this, let \( x_0 \in f^{-1}(U) \), but then what can you tell me about \( f(x_0) \)? and now use the definition of \( U \) open and the fact that \( f \) is continuous to find \( \delta > 0 \) such that \((x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(U)\).

(⇐) If \( \epsilon > 0 \) is given, use \( U = (f(x_0) - \epsilon, f(x_0) + \epsilon) \)

**AP 7:** Suppose \( U = \{U_\alpha\} \) is an open cover of \( f(K) \) and show that \( U \) has a finite sub-cover. To do this, consider the sets \( f^{-1}(U_\alpha) \)
AP 10: Let $x_0 \in \mathbb{R}^k$ and $\epsilon > 0$ be given, and consider $K_n = \overline{B(x_0, \frac{1}{n})}$. Then what is the intersection of $f(K_n)$? Now consider $B = B(f(x_0), \epsilon) = (f(x_0) - \epsilon, f(x_0) + \epsilon)$ and consider $f(K_n) \setminus B$. On the one hand, using the above, show that their intersection is empty. On the other hand, since the sets are nonempty, closed, bounded and decreasing, using the finite intersection property, show that there is $N$ such that $f(K_N) \setminus B = \emptyset$ and conclude that

$$|x - x_0| < \frac{1}{N} \Rightarrow |f(x) - f(x_0)| < \epsilon$$

And conclude that $f$ is continuous.