17.8(b)

Case 1: \( f(x) \leq g(x) \)

In this case \( \min(f, g) = f \), but, since \(-f \geq -g\), we have \( \max(-f, -g) = -f \), and therefore

\[-\max(-f, -g) = -(-f) = f = \min(f, g) \checkmark\]

Case 1: \( g(x) \leq f(x) \)

In this case \( \min(f, g) = g \), but, since \(-g \geq -f\), we have \( \max(-f, -g) = -g \), and therefore

\[-\max(-f, -g) = -(-g) = f = \min(f, g) \checkmark \]

17.9(a)

STEP 1: Scratchwork

\[
|f(x) - f(x_0)| = |f(x) - f(2)| \\
= |x^2 - 4| \\
= |x - 2||x + 2| \\
? \quad < \epsilon
\]
Now if $|x - 2| < 1$, then $-1 < x - 2 < 1$ so $1 < x < 3$, hence $3 < x + 2 < 5$ and therefore $|x + 2| < 5$. Hence

$$|x - 2| |x + 2| \leq 5|x - 2| < \epsilon \Rightarrow |x - 2| < \frac{\epsilon}{5}$$

**STEP 2:** Actual proof

Let $\epsilon > 0$ be given, let $\delta = \min\{1, \frac{\epsilon}{5}\} > 0$ and suppose $|x - 2| < \delta$, then $|x + 2| < 5$ and therefore

$$|f(x) - f(2)| = |x - 2| |x + 2| \leq 5|x - 2| < 5 \left(\frac{\epsilon}{5}\right) = \epsilon$$

Hence $f(x) = x^2$ is continuous at $x_0 = 2$ \hfill $\square$

17.9(b)

**STEP 1:** Scratchwork

$$|f(x) - f(x_0)| = |\sqrt{x}| = \sqrt{x} < \epsilon \Rightarrow |x| < \epsilon^2$$

**STEP 2:** Actual Proof

Let $\epsilon > 0$ be given, let $\delta = \epsilon^2$, then if $|x| < \delta$, then

$$|f(x) - f(x_0)| = |\sqrt{x}| < \sqrt{\epsilon^2} = \epsilon$$

Hence $f(x) = \sqrt{x}$ is continuous at $x_0 = 0$ \hfill $\square$

17.9(c)
**STEP 1:** Scratchwork

\[ |f(x) - f(x_0)| = \left| x \sin \left( \frac{1}{x} \right) \right| = |x| \left| \sin \left( \frac{1}{x} \right) \right| \leq |x| < \epsilon \]

**STEP 2:** Actual Proof

Let \( \epsilon > 0 \) be given, let \( \delta = \epsilon \), then if \( |x| < \delta \), then

\[ |f(x) - f(x_0)| = |x| \left| \sin \left( \frac{1}{x} \right) \right| \leq |x| < \epsilon \sqrt{ } \]

Hence \( f(x) \) is continuous at \( x_0 = 0 \)

\[ \square \]

17.9(d)

**STEP 1:** Scratchwork

\[ |f(x) - f(x_0)| = \left| x^3 - (x_0)^3 \right| = |x - x_0| \left| x^2 + x_0x + (x_0)^2 \right| \leq |x - x_0| \left( |x|^2 + |x_0||x| + |x_0|^2 \right) \]

Now if \( |x - x_0| < 1 \), then

\[ |x| = |x - x_0 + x_0| \leq |x - x_0| + |x_0| < 1 + |x_0| \]
And therefore

\[ |x - x_0| \left( |x|^2 + |x_0| |x| + |x_0|^2 \right) \]
\[ \leq |x - x_0| \left( (1 + |x_0|)^2 + |x_0| (1 + |x_0|) + |x_0|^2 \right) \]
\[ \leq \frac{|x - x_0|^C C}{C} = \epsilon \]

Which gives \( \delta = \frac{\epsilon}{C} \)

**STEP 2:** Actual proof

Let \( \epsilon > 0 \) be given, let \( \delta = \min \{ 1, \frac{\epsilon}{C} \} > 0 \) and suppose \( |x - x_0| < \delta \), then \( |x| \leq |x_0| + 1 \) and therefore

\[ |f(x) - f(x_0)| = |x - x_0| \left( |x|^2 + x_0 x + (x_0)^2 \right) \]
\[ \leq |x - x_0| \left( |x|^2 + |x_0| |x| + |x_0|^2 \right) \]
\[ \leq |x - x_0| C \]
\[ < \left( \frac{\epsilon}{C} \right) C \]
\[ = \epsilon \sqrt{C} \]

Hence \( f(x) = x^3 \) is continuous at \( x_0 \)

\[ 17.10(A) \]

**The Sequence Way:** Let \( (x_n) \) be a sequence of positive numbers converging to 0, for instance \( x_n = \frac{1}{n} \). Then \( x_n \to 0 \), but

\[ f(x_n) = 1 \not\to 0 = f(0) \]
Therefore $f$ is not continuous at $x_0 = 0$.

**The $\epsilon - \delta$ Way:** Let $\epsilon = \frac{1}{2}$, then, if $\delta > 0$, let $x = \text{any positive number such that } |x| < \delta$, for instance $x = \frac{\delta}{2}$, then $|x - 0| < \delta$, but

$$|f(x) - f(0)| = |1 - 0| = 1 < \frac{1}{2} = \epsilon$$

Hence $f$ is not continuous at $x_0 = 0$

---

17.10(b)

Choose $x_n$ such that $\sin\left(\frac{1}{x_n}\right) = 1$, so $\frac{1}{x_n} = \frac{\pi}{2} + 2\pi n$, so

$$x_n = \frac{1}{\frac{\pi}{2} + 2\pi n}$$

Then $x_n \to 0$ but $g(x_n) = \sin\left(\frac{\pi}{2} + 2\pi n\right) = 1 \not\to g(0) = 0$.

Hence $g$ is not continuous at $x_0 = 0$

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17.12(a)

Let $x \in (a, b)$, then since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a sequence $(x_n)$ of rational numbers converging to $x$. But then, on the other hand, $f(x_n) \to f(x)$ since $f$ is continuous. And, on the other hand, since $f(x_n) = 0$ (since $x_n$ is rational), we get $f(x) = 0$.

---

17.12(b)

Consider $h(x) = f(x) - g(x)$. Then $h$ is continuous, being the difference of continuous functions. Moreover, if $r$ is rational, then $h(r) = f(r) - g(r) = 0 - 0 = 0$, therefore by part (a), we have $h(x) = 0$ for
all $x$, and so $f(x) - g(x) = 0$, that is $f(x) = g(x)$ for all $x$ \hfill \Box

17.13(A)

**Case 1:** If $x$ is irrational, then since $\mathbb{Q}$ is dense in $\mathbb{R}$, let $x_n$ be a sequence of rational numbers converging to $x$, but then

$$f(x_n) = 1 \not\to 0 = f(x)$$

Therefore $f$ is discontinuous at irrational $x$

**Case 2:** If $x$ is rational, let $x_n = x + \frac{\sqrt{2}}{n}$. Then $x_n$ is irrational (otherwise $\sqrt{2} = n(x_n - x)$ would be rational), and $x_n \to x$, but

$$f(x_n) = 0 \not\to 1 = f(x)$$

Hence $f$ is discontinuous at rational $x$

17.13(B)

First of all, let’s show that $h$ is continuous at $x = 0$. Let $\epsilon > 0$ be given and let $\delta = \epsilon > 0$, then if $|x| < \delta$, then

$$|h(x) - h(0)| = |h(x)| \leq |x| < \epsilon \sqrt{x}$$

(Here we used $|h(x)| \leq |x|$, which you can show by cases)

Therefore $h$ is continuous at $x$

Now if $x \neq 0$ is irrational, then, since $\mathbb{Q}$ is dense in $\mathbb{R}$, let $x_n$ be a sequence of rational numbers converging to $x$, but then

$$h(x_n) = x_n \to x \neq 0 = h(x)$$
Hence \( h \) is discontinuous at irrational \( x \)

And if \( x \neq 0 \) is rational, then let \( x_n = x + \sqrt{2} \frac{n}{n} \), which is a sequence of irrational numbers converging to \( x \), but then

\[
h(x_n) = 0 \rightarrow 0 \neq x = h(x)
\]

Hence \( h \) is discontinuous at rational \( x \)

17.14

**STEP 1**: Suppose \( x_0 \) is rational, and let’s show that \( f \) is discontinuous at \( x_0 \). let \( x_n = x + \sqrt{2} \frac{n}{n} \). Then \( x_n \) is a sequence of irrational numbers converging to \( x_0 \), but then

\[
f(x_n) = 0 \not\rightarrow f(x_0) \neq 0
\]

And therefore \( f \) is discontinuous at \( x_0 \).

And if \( x_0 = 0 \), then let \( x_n = \frac{1}{n} \rightarrow 0 \), then \( x_n \rightarrow x_0 \) but then

\[
f(x_n) = n \rightarrow \infty \neq 0 = f(x_0)
\]

Therefore \( f \) is also discontinuous at 0

Hence \( f \) is discontinuous at all the rational numbers

**STEP 2**: Suppose \( x_0 \) is irrational, and let \( \epsilon > 0 \) be given.

Then let \( N \) be such that \( \frac{1}{N} < \epsilon \)

Now let \( \delta \) be so small such that there are no integers in \((x_0 - \delta, x_0 + \delta)\) and no fractions with denominator 2, no fractions with denominator
3, . . . and no fractions with denominator $\frac{1}{N}$.

Then, if $|x - x_0| < \delta$ and $x$ is irrational, then

$$|f(x) - f(x_0)| = |0 - 0| = 0 < \epsilon \checkmark$$

And if $x = \frac{p}{q}$ is rational (where $p$ and $q$ have no common factors and $q > 0$), then by the above we must have $q > N$, and so

$$|f(x) - f(x_0)| = \left| \frac{1}{q} - 0 \right| = \frac{1}{q} < \frac{1}{N} < \epsilon$$

Therefore $f$ is continuous at $x_0 \checkmark$

And hence $f$ is continuous at all irrational numbers and discontinuous at all rational numbers. \qed