AP 1

(a) Let $\epsilon > 0$ be given, let $\delta = \epsilon$, then if $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| = ||x| - |x_0|| \leq |x - x_0| < \epsilon \checkmark$$

Hence $f(x) = |x|$ is continuous

(b) **STEP 1:** Scratchwork

$$|f(x) - f(x_0)| = \left| \frac{1}{x} - \frac{1}{x_0} \right| = \left| \frac{x_0 - x}{xx_0} \right| = \frac{|x - x_0|}{|x_0||x|}$$

Now if $|x - x_0| < \frac{|x_0|}{2}$, then

$$|x - x_0| \geq ||x| - |x_0|| \geq - (|x| - |x_0|) = |x_0| - |x|$$

And therefore

$$|x_0| - |x| < \frac{|x_0|}{2} \Rightarrow |x| > |x_0| - \frac{|x_0|}{2} = \frac{|x_0|}{2}$$
Hence \( \frac{1}{|x|} < \frac{2}{|x_0|} \), and therefore:

\[
|f(x) - f(x_0)| = \left( \frac{|x - x_0|}{|x_0|} \right) \left( \frac{1}{|x|} \right) \\
\leq \left( \frac{|x - x_0|}{|x_0|} \right) \left( \frac{2}{|x_0|} \right) \\
= |x - x_0| \left( \frac{2}{|x_0|^2} \right) \\
< \epsilon
\]

Which gives \( |x - x_0| = \frac{\epsilon |x_0|^2}{2} \)

**STEP 2:** Actual Proof

Let \( \epsilon > 0 \) be given, let \( \delta = \min \left\{ \frac{|x_0|}{2}, \frac{\epsilon |x_0|^2}{2} \right\} \), then if \( |x - x_0| < \delta \), then

\[
|f(x) - f(x_0)| = \left( \frac{|x - x_0|}{|x_0|} \right) \left( \frac{1}{|x|} \right) \\
\leq \left( \frac{|x - x_0|}{|x_0|} \right) \left( \frac{2}{|x_0|} \right) \\
= |x - x_0| \left( \frac{2}{|x_0|^2} \right) \\
< \left( \frac{\epsilon |x_0|^2}{2} \right) \left( \frac{-2}{|x_0|^2} \right) \\
= \epsilon
\]

Hence \( f \) is continuous at \( x_0 \)
(c) **STEP 1:** Scratchwork

\[ |f(x) - f(x_0)| = |\sqrt{x} - \sqrt{x_0}| \]

\[ = |(\sqrt{x} - \sqrt{x_0}) \left( \frac{\sqrt{x} + \sqrt{x_0}}{\sqrt{x} + \sqrt{x_0}} \right)| \]

\[ = |(\sqrt{x})^2 - (\sqrt{x_0})^2| \left| \frac{1}{\sqrt{x} + \sqrt{x_0}} \right| \]

\[ = |x - x_0| \left( \frac{1}{\sqrt{x} + \sqrt{x_0}} \right) \]

\[ \leq |x - x_0| \left( \frac{1}{\sqrt{x_0}} \right) \]

\[ < \frac{1}{\sqrt{x_0}} \epsilon \]

Which gives \(|x - x_0| < (\sqrt{x_0}) \epsilon\)

**STEP 2:** Actual Proof

Let \(\epsilon > 0\) be given, let \(\delta = (\sqrt{x_0}) \epsilon\), then if \(|x - x_0| < \delta\), then

\[ |f(x) - f(x_0)| = |x - x_0| \left( \frac{1}{\sqrt{x} + \sqrt{x_0}} \right) \]

\[ \leq |x - x_0| \left( \frac{1}{\sqrt{x_0}} \right) \]

\[ < \left( \frac{1}{\sqrt{x_0}} \right) \epsilon \]

\[ = \epsilon \]

Hence \(f\) is continuous at \(x_0\)
AP 2

Let $\epsilon > 0$ be given, let $\delta = \frac{\epsilon}{C}$, then if $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| \leq C |x - x_0| < C \left( \frac{\epsilon}{C} \right) = \epsilon \sqrt{\epsilon}$$

Hence $f$ is continuous at $x_0$, and hence continuous \( \square \)

AP 3

(a)\[ x \in f^{-1}((7, 10)) \iff f(x) \in (7, 10) \]
\[ \iff 7 < 3x + 7 < 10 \]
\[ \iff 0 < 3x < 3 \]
\[ \iff 0 < x < 1 \]

Hence $f^{-1}(U) = (0, 1)$

(b)\[ x \in f^{-1}((-1, 4)) \iff f(x) \in (-1, 4) \]
\[ \iff -1 < x^2 < 4 \]
\[ \iff -2 < x < 2 \]

Hence $f^{-1}(U) = (-2, 2)$

(c)\[ x \in f^{-1}((0, 1)) \iff f(x) \in (0, 1) \]
\[ \iff 0 < \sin(x) < \pi \]
\[ \iff x \in (2m\pi, (2m + 1)\pi), m \in \mathbb{Z} \]

Hence
\[ f^{-1}((0,1)) = \bigcup_{m \in \mathbb{Z}} (2m\pi, (2m+1)\pi) = \cdots \cup (-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cdots \]

**AP 4**

(a) \[ x \in (g \circ f)^{-1}(U) \iff (g \circ f)(x) \in U \]
\[ \iff g(f(x)) \in U \]
\[ \iff f(x) \in g^{-1}(U) \]
\[ \iff x \in f^{-1}(g^{-1}(U)) \]

(b) Suppose \( U \) is open, then since \( g \) is continuous, \( g^{-1}(U) \) is open, and hence, since \( f \) is continuous, \( f^{-1}(g^{-1}(U)) \) is open, and therefore

\[ (g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \text{ is open} \checkmark \]

Hence \( g \circ f \) is continuous \( \square \)

**AP 5**

(\( \Rightarrow \)) Suppose \( f \) is continuous and let \( U \) be open. We want to show \( f^{-1}(U) \) is open.

Let \( x_0 \in f^{-1}(U) \). Then, by definition \( f(x_0) \in U \). Since \( U \) is open, there is \( \epsilon > 0 \) such that \( (f(x_0) - \epsilon, f(x_0) + \epsilon) \subseteq U \)
However, since $f$ is continuous, there is $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

**Claim:** $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(U)$

(Then we’re done because this shows $f^{-1}(U)$ is open)

Suppose $x \in (x_0 - \delta, x_0 + \delta)$, then $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon) \subseteq U$, and so $f(x) \in U$ and so $x \in f^{-1}(U)$

$(\Leftarrow)$ Suppose $f^{-1}(U)$ is open whenever $U$ is open, and let’s show $f$ is continuous.

Fix $x_0$. Let $\epsilon > 0$ be given, then notice that $U = (f(x_0) - \epsilon, f(x_0) + \epsilon)$ is open, and therefore, by assumption, $f^{-1}(U)$ is open.

Moreover, since $f(x_0) \in U$, $x_0 \in f^{-1}(U)$ (which is open), and therefore, by definition, there is $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(U)$

But then, with that $\delta$ if $|x - x_0| < \delta$, then $x \in (x_0 - \delta, x_0 + \delta)$ and so $x \in f^{-1}(U)$, which means $f(x) \in U = (f(x_0) - \epsilon, f(x_0) + \epsilon)$, so $|f(x) - f(x_0)| < \epsilon$, and so $f$ is continuous at $x_0$, and hence continuous.

$\square$

**AP 6**

(a)

$x \in f^{-1}(A \cup B) \iff f(x) \in A \cup B$

$\iff (f(x) \in A) \text{ or } (f(x) \in B)$

$\iff (x \in f^{-1}(A)) \text{ or } (x \in f^{-1}(B))$

$\iff x \in f^{-1}(A) \cup f^{-1}(B)$
(b) \[ x \in f^{-1}(A \cap B) \iff f(x) \in A \cap B \]
\[ \iff (f(x) \in A) \text{ and } (f(x) \in B) \]
\[ \iff (x \in f^{-1}(A)) \text{ and } (x \in f^{-1}(B)) \]
\[ \iff x \in f^{-1}(A) \cap f^{-1}(B) \]

(c) \[ x \in f^{-1}(A^c) \iff f(x) \in A^c \]
\[ \iff f(x) \notin A \]
\[ \iff \text{Not } (f(x) \in A) \]
\[ \iff \text{Not } (x \in f^{-1}(A)) \]
\[ \iff x \notin f^{-1}(A) \]
\[ \iff x \in (f^{-1}(A))^c \]

AP 7

(a) **STEP 1:** Let \( \mathcal{U} = \{ U_\alpha \} \) be an open cover of \( f(K) \), and consider \( \mathcal{U}' = \{ f^{-1}(U_\alpha) \} \).

**STEP 2:** Then, since \( U_\alpha \) is open and \( f \) is continuous, \( f^{-1}(U_\alpha) \) is open.

Moreover, by an analog of AP6(b), we have
\[ \bigcup_{\alpha} f^{-1}(U_\alpha) = f^{-1} \left( \bigcup_{\alpha} U_\alpha \right) \]

And, since \( \mathcal{U} \) covers \( f(K) \), we have \( K \subseteq \bigcup_{\alpha} U_\alpha \) and so \( f^{-1}(\bigcup_{\alpha} U_\alpha) \supseteq f^{-1}(f(K)) \)
And finally $K \subseteq f^{-1}(f(K))$ since if $x \in K$, then $f(x) \in f(K)$ and so $x \in f^{-1}(f(K))$.

Therefore, combining everything, we get

$$\bigcup_{\alpha} f^{-1}(U_\alpha) \supset K$$

**STEP 3:** So $U'$ covers $K$. But since $K$ is compact, there is a finite sub-cover

$$V' = \{ f^{-1}(U_{n_1}), \ldots, f^{-1}(U_{n_N}) \}$$

**STEP 4:**

Claim:

$$V =: \{ U_{n_1}, \ldots, U_{n_N} \}$$

Covers $K$

(Then we would be done because we found a finite sub-cover of $U$)

But if $y \in f(K)$, then $y = f(x)$ for some $x \in K$, but since $V$ covers $K$, $x \in f^{-1}(U_{n_k})$ for some $k$, and so $y = f(x) \in U_{n_k} \in V$.

\[\checkmark\]

(b) No since $[0,1]$ is compact, and so $f([0,1])$ would be compact, but $f([0,1]) = (0,1)$, which is not compact
(c) Since \([a, b]\) is compact and \(f\) is continuous, \(f([a, b])\) is compact, and therefore bounded, which means that \(f\) is bounded (that is there is \(M > 0\) such that \(|f(x)| \leq M\) for all \(x \in [a, b]\))

AP 8

(⇒) Let \(\epsilon > 0\) be given, then there is \(\delta > 0\) such that if \(d(x, x_0) < \delta\), then \(d'(f(x), f(x_0)) < \epsilon\).

But, with that same \(\delta\), if \(d(x, x_0) < \delta\), then for each \(j\),

\[
|f_j(x) - f_j(x_0)| = \sqrt{(f_j(x) - f_j(x_0))^2} \\
\leq \sqrt{\sum_{j=1}^{k} (f_j(x) - f_j(x_0))^2} \\
< \epsilon \sqrt{k}
\]

Hence \(f_j\) is continuous.

(⇐) Let \(\epsilon > 0\) be given, then for each \(j\), there is \(\delta_j > 0\) such that if \(d(x, x_0) < \delta_j\), then \(|f_j(x) - f_j(x_0)| < \frac{\epsilon}{\sqrt{k}}\)

Let \(\delta = \min \{\delta_1, \ldots, \delta_k\} > 0\), then if \(d(x, x_0) < \delta\), then
$$d(f(x), f(x_0)) = \sqrt{\sum_{j=1}^{k} (f_j(x) - f_j(x_0))^2}$$

$$< \sqrt{\sum_{j=1}^{k} \left( \frac{\epsilon}{\sqrt{k}} \right)^2}$$

$$= \sqrt{\sum_{j=1}^{k} \frac{\epsilon^2}{k}}$$

$$= \sqrt{k \left( \frac{\epsilon^2}{k} \right)}$$

$$= \sqrt{\epsilon^2}$$

$$= \epsilon \sqrt{k}$$

Hence $f$ is continuous

AP 9

Let $\epsilon > 0$ be given, let $\delta = \frac{1}{2}$, then if $d(x, x_0) < \delta = \frac{1}{2} < 1$, then $x = x_0$,

and therefore

$$d'(f(x), f(x_0)) = d'(f(x_0), f(x_0)) = 0 < \epsilon \sqrt{k}$$

Hence any $f$ is continuous

AP 10

**STEP 1:** Fix $x_0 \in \mathbb{R}^k$ and let $\epsilon > 0$ be given. Let $K_n = B(x_0, \frac{1}{n})$, notice that the $K_n$ are decreasing, and therefore, by (2), we have
\[
\cap_{n=1}^{\infty} f(K_n) = f \left( \cap_{n=1}^{\infty} K_n \right) = f(\{x_0\}) = \{f(x_0)\}
\]

**STEP 2:** Let \( B = B(f(x_0), \epsilon) = (f(x_0) - \epsilon, f(x_0) + \epsilon) \).

Then, first of all
\[
\cap (f(K_n) \setminus B) = \left( \cap f(K_n) \right) \cap B^c = \{f(x_0)\} \setminus B = \emptyset
\]
(because \( f(x_0) \) is in \( B \))

On the other hand, since \( K_n \) is compact, by (1), \( f(K_n) \) is compact and hence closed, and so \( f(K_n) \setminus B = f(K_n) \cap B^c \) is closed. And since the \( K_n \) are decreasing, the \( f(K_n) \) are decreasing, and so is \( f(K_n) \setminus B \).

Now if for all \( n \), \( (f(K_n) \setminus B) \neq \emptyset \), then by the finite intersection property we would have \( \cap (f(K_n) \setminus B) \neq \emptyset \), which contradicts the above.

Therefore, for some \( N \), \( f(K_N) \setminus B = f(K_N) \cap B^c = \emptyset \).

**STEP 3:** But this implies that \( f(K_N) \subseteq B \), and therefore, if \( |x - x_0| < \frac{1}{N} \), then \( x \in B(x_0, \frac{1}{N}) = K_N \), and so \( f(x) \in f(K_N) \subseteq B = B(f(x_0), \epsilon) \), meaning \( |f(x) - f(x_0)| < \epsilon \). In other words
\[
|x - x_0| < \frac{1}{N} \Rightarrow |f(x) - f(x_0)| < \epsilon
\]

**STEP 4:** Now given \( \epsilon > 0 \), let \( \delta < \frac{1}{N} \) as above, then if \( |x - x_0| < \delta < \frac{1}{N} \), then \( |f(x) - f(x_0)| < \epsilon \), and therefore \( f \) is continuous at \( x_0 \), and hence is continuous. \( \Box \)