HOMEWORK 6 – SELECTED BOOK SOLUTIONS

12.4

STEP 1: Let $N$ be given. First, let’s show that

$$\sup \{s_n + t_n \mid n > N\} \leq \sup \{s_n \mid n > N\} + \sup \{t_n \mid n > N\}$$

But if $n > N$, then by definition of sup,

$$s_n \leq \sup \{s_n \mid n > N\}$$

And

$$t_n \leq \sup \{t_n \mid n > N\}$$

So adding both sides, we get

$$s_n + t_n \leq \sup \{s_n \mid n > N\} + \sup \{t_n \mid n > N\}$$

And taking the sup over all $n > N$, we get:

$$\sup \{s_n + t_n \mid n > N\} \leq \sup \{s_n \mid n > N\} + \sup \{t_n \mid n > N\}$$

STEP 2: Taking $\lim_{N \to \infty}$ in the above identity, we get:

Date: Due: Thursday, May 7, 2020.
\[
\limsup_{n \to \infty} s_n + t_n \overset{\text{DEF}}{=} \lim_{N \to \infty} \sup \{s_n + t_n \mid n > N\}
\]

**STEP 1:** Let \( N \) be given, then for all \( n > N \), by definition of sup

\[
s_n \leq \sup \{s_n \mid n > N\} \quad t_n \leq \sup \{t_n \mid n > N\}
\]

Therefore, since \( s_n \geq 0 \) and \( t_n \geq 0 \), we get:

\[
s_n t_n \leq (\sup \{s_n \mid n > N\}) (\sup \{t_n \mid n > N\})
\]

Now taking the sup over \( n > N \), we get:

\[
\sup \{s_n t_n \mid n > N\} \leq \sup \{s_n \mid n > N\} \sup \{t_n \mid n > N\}
\]

**STEP 2:** Now taking the limit as \( N \to \infty \) in the above, we get

\[
\limsup_{n \to \infty} s_n t_n \overset{\text{DEF}}{=} \lim_{N \to \infty} \sup \{s_n t_n \mid n > N\}
\]

\[
\leq \lim_{N \to \infty} \sup \{s_n \mid n > N\} \sup \{t_n \mid n > N\}
\]

\[
= \lim_{N \to \infty} \sup \{s_n \mid n > N\} \lim_{N \to \infty} \sup \{t_n \mid n > N\}
\]

\[
= \left( \lim_{n \to \infty} s_n \right) \left( \lim_{n \to \infty} t_n \right)
\]

\(\checkmark\)
12.12(a)

The middle inequality follows because \( \liminf \leq \limsup \), and the first inequality is similar to the third, so let's just show the third inequality, which is:

\[
\limsup_{n \to \infty} \sigma_n \leq \limsup_{n \to \infty} s_n
\]

**STEP 1:** Let \( N \) be given, and suppose \( M > N \). Let's show that

\[
\sup \{ \sigma_n \mid n > M \} \leq \frac{1}{M} (s_1 + \cdots + s_N) + \sup \{ s_n \mid n > N \}
\]

Notice that for all \( n > M \), we have

\[
\sigma_n = \frac{1}{n} (s_1 + \cdots + s_n) \\
\leq \frac{s_1 + \cdots + s_N}{n} + \frac{s_{N+1} + \cdots + s_n}{n} \\
\leq \frac{s_1 + \cdots + s_N}{n} + \frac{n - N}{n} \sup \{ s_n \mid n > N \}
\]

Notice there are \( n - (N + 1) + 1 = n - N \) terms in \( s_{N+1} + \cdots + s_n \). Moreover, by definition of sup, each term \( s_{N+1}, s_{N+2}, \ldots, s_n \) is \( \leq \sup \{ s_n \mid n > N \} \), and therefore we get:

\[
\sigma_n \leq \frac{s_1 + \cdots + s_N}{n} + \frac{n - N}{n} \sup \{ s_n \mid n > N \} \\
\leq \frac{s_1 + \cdots + s_N}{M} + \sup \{ s_n \mid n > N \}
\]

(Here we used \( n > M \Rightarrow \frac{1}{n} < \frac{1}{M} \))

Since the right-hand-side doesn’t depend on \( n \), we therefore get:
\begin{align*}
\sup \{\sigma_n \mid n > M\} \leq \frac{s_1 + \cdots + s_N}{M} + \sup \{s_n \mid n > N\} \checkmark
\end{align*}

**STEP 2:** Now be careful: *First* let \( M \to \infty \) to get:

\begin{align*}
\limsup_{n \to \infty} \sigma_n &= \lim_{M \to \infty} \sup \{\sigma_n \mid n > M\} \\
&\leq \lim_{M \to \infty} \frac{1}{M} (s_1 + s_2 + \cdots + s_N) + \sup \{s_n \mid n > N\} \\
&= \lim_{M \to \infty} \frac{1}{M} (s_1 + s_2 + \cdots + s_N) + \lim_{M \to \infty} \sup \{s_n \mid n > N\}
\end{align*}

But since \( N \) is fixed, \( s_1 + s_2 + \cdots + s_N \) doesn’t depend on \( M \), and so

\[ \lim_{M \to \infty} \frac{1}{M} (s_1 + s_2 + \cdots + s_N) = 0 \]

And moreover \( \{s_n \mid n > N\} \) is constant with respect to \( M \), and so

\[ \lim_{M \to \infty} \sup \{s_n \mid n > N\} = \sup \{s_n \mid n > N\} \]

Therefore we get:

\[ \limsup_{n \to \infty} \sigma_n \leq \sup \{s_n \mid n > N\} \]

But since the left-hand-side doesn’t depend on \( N \), we can let \( N \) go to \( \infty \) to get:

\[ \limsup_{n \to \infty} \sigma_n \leq \lim_{N \to \infty} \sup \{s_n \mid n > N\} = \limsup_{N \to \infty} s_n \checkmark \]

12.12(b)

If \( \lim_{n \to \infty} s_n = L \), then we get:
\[ L = \liminf_{n \to \infty} s_n \leq \liminf_{n \to \infty} \sigma_n \leq \limsup_{n \to \infty} \sigma_n \leq \limsup_{n \to \infty} s_n = L \]

Therefore

\[ \liminf_{n \to \infty} s_n = \limsup_{n \to \infty} s_n = L \]

So by the lim sup squeeze theorem, we have

\[ \lim_{n \to \infty} s_n = L \checkmark \]

12.12(c)

Let \((s_n) = (-1)^n\), then \(\lim_{n \to \infty} s_n \) doesn’t exist, but

\[
\sigma_n = \frac{1}{n} (s_1 + s_2 + \cdots + s_n) \\
= \frac{1}{n} (-1 + 1 - 1 + 1 - 1 + \cdots + (-1)^n) \\
= \frac{a_n}{n}
\]

Where \(a_n\) is either 0 (if \(n\) is even) or 1 if \(n\) is odd, but then by the squeeze theorem, we get that \(\sigma_n \to 0\) as \(n \to \infty\) \(\checkmark\)

13.4

(a) Let \(\{E_\alpha\}\) be any collection of open subsets (where \(\alpha \in I\) for some indexing set \(I\))

Let \(x \in \bigcup E_\alpha\), then \(x \in E_\alpha\) for some \(\alpha\).
But then since $E_{\alpha}$ is open, there is some $r$ with $B(x, r) \subseteq E_{\alpha}$.

But then since $E_{\alpha} \subseteq \bigcup E_{\alpha}$, we get: $B(x, r) \subseteq \bigcup E_{\alpha}$ for some $r$.

(b) Let $E_1, E_2, \ldots, E_n$ be a finite collection of open subsets.

Let $x \in \bigcap_{i=1}^n E_i$. Then for all $i = 1, 2, \ldots, n$, $x \in E_i$.

But since $E_i$ is open, there is $r_i > 0$ with $B(x, r_i) \subseteq E_i$.

Now let $r = \min \{r_1, r_2, \ldots, r_n\} > 0$.

**Claim:** $B(x, r) \subseteq \bigcap_{i=1}^n E_i$.

If $y \in B(x, r)$, then by definition of $r$, we have $y \in B(x, r_i)$ for all $i$. And so $y \in B(x, r_i) \subseteq E_i$, and therefore $y \in E_i$, for all $i$, and so $y \in \bigcap_{i=1}^n E_i$.

Therefore there is some $r > 0$ with $B(x, r) \subseteq \bigcap_{i=1}^n E_i$.

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13.7

**STEP 1:** Let $U \subseteq \mathbb{R}$ be an open subset of $\mathbb{R}$. Let $(q_n)$ be an enumeration of the rational numbers in $U$. Let

$$a_n = \inf \{a \in \mathbb{R} \mid (a, q_n] \subseteq U\} \quad b_n = \sup \{b \in \mathbb{R} \mid [q_n, b) \subseteq U\}$$
STEP 2:

Claim: For each $n$, $(a_n, b_n) \subseteq U$

Proof: Suppose $x \in (a_n, b_n)$. Then:

$$x > a_n = \inf \{a \in \mathbb{R} \mid (a, q_n) \subseteq U\}$$

So there is $a < x$ with $(a, q_n) \subseteq U$.

Similarly $x < b_n = \sup \{b \in \mathbb{R} \mid [q_n, b) \subseteq U\}$, so there is $b > x$ with $[q_n, b) \subseteq U$.

But then $(a, b) = (a, q_n) \cup [q_n, b) \subseteq U \cup U = U$

Since $a < x < b$, we have $x \in (a, b) \subseteq U$, so $x \in U \checkmark$

STEP 3:

Claim: $U = \bigcup_{n=1}^{\infty} (a_n, b_n)$

Proof: First of all, if $x \in \bigcup_{n=1}^{\infty} (a_n, b_n)$, then $x \in (a_n, b_n) \subseteq U$ for some $n$, so $x \in U \checkmark$

If $x \in U$, then since $U$ is open, there is $r > 0$ with $(x-r, x+r) \subseteq U$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a rational number $q_n \in (x-r, x+r) \subseteq U$.

But then $(x-r, q_n] \subseteq (x-r, x+r) \subseteq U$, so by definition of $a_n$ (as an inf), $a_n \leq x-r$, so $x \geq r + a_n > a_n$. And similarly $[q_n, x+r) \subseteq U$, so by definition of $b_n$, $b_n \geq x+r$, so $x \leq b_n - r < b_n$. And finally we get $a_n < x < b_n$ so $x \in (a_n, b_n)$. $\checkmark$
STEP 4:

Claim: \((a_n, b_n) \cap (a_m, b_m) \neq \emptyset \Rightarrow (a_n, b_n) = (a_m, b_m)\)

Assume \((a_m, b_m) \cap (a_n, b_n) \neq \emptyset\). Since we either have \(a_m \leq a_n\) or \(a_n \leq a_m\), so WLOG, assume \(a_m \leq a_n\).

If \(b_m > a_n\), then \(b_n \leq a_n\) so \(a_m < b_m \leq a_n < b_n\) and so \((a_m, b_m) \cap (a_n, b_n) = \emptyset \Rightarrow \Leftarrow\). Hence \(b_m \leq a_n\)

Case 1: \(b_n \geq b_m\)

Then \((a_m, b_m) \cup (a_n, b_n) = (a_m, b_n) \subseteq U\), so \([q_m, b_n] \subseteq U\) and so by definition of \(b_m\) (as a sup), we have \(b_m \geq b_n\). And so \(b_n = b_m\).

Case 2: \(b_m \geq b_n\)

In this case \([q_n, b_m] \subseteq U\) and so by definition of \(b_n\) we have \(b_n \geq b_n\), so \(b_m = b_n\).

In any case, we have \(b_n = b_m\).

Finally, since \(a_m \leq a_n\), we have \((a_m, q_n] \subseteq U\) and so by definition of \(a_n\) (as an inf), we have \(a_n \leq a_m\) and therefore \(a_n = a_m\)

In conclusion, we get \(a_n = a_m\) and \(b_n = b_m\) so \((a_n, b_n) = (a_m, b_m)\)

STEP 5: The previous step is saying that the intervals \((a_n, b_n)\) are either equal or disjoint.

Now we know \(U = \bigcup_{n=1}^{\infty} (a_n, b_n)\)
Construct a (possibly finite) subsequence \((a_{n_k}, b_{n_k})\) as follows:

**Base Case:** \((a_{n_1}, b_{n_1}) = (a_1, b_1)\)

**Inductive Step:** Suppose \((a_{n_1}, b_{n_1}), \ldots, (a_{n_k}, b_{n_k})\) (disjoint) have been selected

**Case 1:** \((a_{(n_k)+1}, b_{(n_k)+1})\) is different from the previous \((a_{n_i}, b_{n_i})\), then let \((a_{n_k+1}, b_{n_k+1}) = (a_{(n_k)+1}, b_{(n_k)+1})\)

**Case 2:** \((a_{(n_k)+1}, b_{(n_k)+1})\) is equal to some of the previous \((a_{n_i}, b_{n_i})\). Then ignore it and move on to the next interval \((a_{(n_k)+2}, b_{(n_k)+2})\) and repeat. If all the remaining intervals are the same (or if there are none left) then stop.

In that case, you obtain a subsequence \((a_{n_k}, b_{n_k})\) such that all \((a_{n_k}, b_{n_k})\) are disjoint and

\[
U = \bigcup_{k=1}^{N} (a_{n_k}, b_{n_k})
\]

(Where \(N\) is either finite or \(N = \infty\))

13.10

(a) Let \(E\) be the set in question. Suppose \(\frac{1}{n}\) is in \(E^\circ\). Then there is \(r > 0\) such that \(B\left(\frac{1}{n}, r\right) = \left(\frac{1}{n} - r, \frac{1}{n} + r\right) \subseteq E\) But simply let \(y\) be any irrational number between \(\frac{1}{n} - r\) and \(\frac{1}{n} + r\), then \(y\) cannot in the set, since every element in \(E\) is of the form \(\frac{1}{n}\) and therefore rational. Therefore \(B\left(\frac{1}{n}, r\right) \not\subseteq E\), so \(\frac{1}{n}\) cannot be in \(E^\circ\)
(b) Suppose $x$ is in $\mathbb{Q}^\circ$. Then there is $r > 0$ such that $B(x, r) = (x - r, x + r) \subseteq \mathbb{Q}$. But simply let $y$ be any irrational number between $x - r$ and $x + r$, then $y \notin \mathbb{Q}$. Therefore $B \left( \frac{1}{n}, r \right) \subsetneq \mathbb{Q}$, so $x \notin \mathbb{Q}^\circ$.

(c) Suppose $x \in F^\circ$. Then there is $r > 0$ with $B(x, r) = (x - r, x + r) \subseteq F$. Then the length of $(x - r, x + r)$ is less than or equal to the length of $F$. But the length of $(x - r, x + r)$ is $2r$ whereas the length of $F$ is 0, so $2r \leq 0$, which is a contradiction.

\textbf{12.14(A)}

Let $s_n = n!$, then:

$$\left| \frac{s_{n+1}}{s_n} \right| = \frac{(n + 1)!}{n!} = n + 1$$

Therefore:

$$\lim \inf_{n \to \infty} \left| \frac{s_{n+1}}{s_n} \right| = \lim_{n \to \infty} n + 1 = \infty$$

So by the Pre-Root test, we have:

$$\lim \inf_{n \to \infty} (n!)^{\frac{1}{n}} = \lim \inf_{n \to \infty} |s_n|^{\frac{1}{n}} \geq \lim \inf_{n \to \infty} \left| \frac{s_{n+1}}{s_n} \right| = \infty$$

Therefore

$$\lim_{n \to \infty} (n!)^{\frac{1}{n}} = \infty$$

\textbf{12.14(B)}

First of all,
\[
\frac{1}{n} \left( \frac{n!}{n^n} \right)^{\frac{1}{n}} = \left( \frac{n!}{n^n} \right)^{\frac{1}{n}}
\]

So let \( s_n = \frac{n!}{n^n} \), then:

\[
\left| \frac{s_{n+1}}{s_n} \right| = \frac{(n+1)!}{n!} \frac{n^n}{n^{n+1}} = (n+1) \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^{n+1}}
\]

\[
= \left( \frac{n}{n+1} \right)^n
\]

\[
= \left( \frac{n}{n \left(1+\frac{1}{n} \right)} \right)^n
\]

\[
= \left( \frac{1}{1+\frac{1}{n}} \right)^n
\]

\[
\rightarrow \frac{1}{e}
\]

Therefore:

\[
\lim_{n \to \infty} \left| \frac{s_{n+1}}{s_n} \right| = \frac{1}{e}
\]

So by the Corollary of the Pre-Root Test, we have
\[
\lim_{n \to \infty} \frac{1}{n} \left( \frac{1}{n} \right)^{\frac{1}{n}} = \lim_{n \to \infty} |s_n|^{\frac{1}{n}} = \frac{1}{e}
\]