Midterm 1 takes place on **Friday, April 24, 2020** from 11 am to 12 pm. It is a closed book and closed notes exam, and counts for 20% of your grade. You will take the exam on Canvas, the same way you take your quizzes. You can find instructions on how to online exams on my website. Note that the 60 mins to take the exam includes the time to upload your answers. I will try to write a 45-50 min exam, so that you have at least 10 mins to scan and upload everything. There will be 4 problems in total, so you should roughly spend 10-12 mins per problem.

Midterm 1 covers everything up to and including section 9, with the exception of section 6 (Cuts), which won’t be on the exam. The most important sections to focus on are sections 4, 8, and 9, although I could ask you about the other sections as well.

**Proofs you should know**

Know how to prove the following theorems. I could theory ask you to reprove any of the below (or variations thereof):

1. Triangle Inequality and Reverse Triangle Inequality
2. \( \inf(S) = -\sup(-S) \) and deduce the Greatest lower bound property from that (page 6-7 of Lecture 5 Notes)
3. Archimedean Property
(4) Limits are Unique

(5) Any of the 10 Examples of Limits in Lectures 7-9 (see sections 8 and 9 below)

(6) The fact with sequences bounded away from 0 (see page 6-9 of Lecture 8)

(7) The Squeeze Theorem (Problem 8.5a)

(8) $(s_n)$ converges $\Rightarrow (s_n)$ is bounded

(9) Limit laws such as $s_n + t_n \to s + t$ or $s_n t_n \to st$ or $\frac{t_n}{s_n} \to \frac{t}{s}$

(10) Infinite Limit Laws such as if $s_n \to \infty$ and $t_n \to \infty$, then $s_n t_n \to \infty$ (page 11 of Lecture 9) if $s_n \to \infty$ and $t_n \geq m$ for some $m$, then $s_n + t_n \to \infty$ (Problem 9.11(c))

(11) The Duality Formula (last section in Lecture 9)

**Concepts you should know**

Know how to define the following concepts. I could in theory ask you to define them on the exam.

(1) $x$ is algebraic

(2) Rational Roots Theorem

(3) Triangle Inequality and Reverse Triangle Inequality

(4) max$(S)$ and min$(S)$

(5) $M$ is an upper bound for $S$, $m$ is a lower bound for $S$
(6) sup(S) = M, inf(S) = m

(7) Least Upper Bound Property

(8) sup(S) = \infty, \inf(S) = -\infty

(9) Archimedean Property

(10) \mathbb{Q} is dense in \mathbb{R}.

(11) (s_n) converges to s

(12) The Squeeze Theorem

(13) The Binomial Theorem

(14) s_n \to \infty, s_n \to -\infty

(15) The Duality Formula (last section of Lecture 9)

1. **Section 1: The set \mathbb{N} of natural numbers**
   - You **DON’T** need to know Peano’s Axioms for \mathbb{N}, but know how to use them. Same goes for the Induction Axiom

   - Although I won’t specifically ask about it, understand the Mini Analysis Proof given in Lecture 1, it gives a good taste of what analysis proofs look like

   - **Know how to prove a statement by induction.** I could very well ask you a non-analysis related question that uses induction
• Example 1, Example 3, Problems 1.9, AP1, and AP2 in HW1 are all excellent practice problems with induction. If you’re feeling adventurous, you could even try problem 1.12

• Also check out AP3 in HW1, it’s a good “Find a counterexample” exercise

2. Section 2: The set \( \mathbb{Q} \) of rational numbers

• Know how to show that \( \sqrt{2} \) is irrational. It’s a classical proof that is hopefully familiar to you from previous courses.

• Define what it means for a number to be algebraic

• Show that a number like \( \sqrt{3 - 1} \) is algebraic

• Know the statement of the rational roots theorem, but IGNORE its proof. This is not an algebra class, hehe

• Know how to use the rational roots theorem. In particular, there are 2 important applications: Finding a rational root of a polynomial (Example 1 in Lecture 2) and showing that a number is irrational (Example 2 in Lecture 2 and Examples 2 – 6 in the book). Notice in particular the useful shortcut at the end of Example 6.

• You can ignore AP4-AP6 in HW1. In particular, I won’t ask about equivalence relations on this midterm
3. **Section 3: The set $\mathbb{R}$ of real numbers**

- Do **NOT** memorize the axioms of a field, but know how to use them. In particular, know how to prove all the properties listed in the first Theorem in Lecture 3.

- Similarly, do **NOT** memorize the axioms of an ordered field, but know how to use them and how to prove all the properties in the second Theorem in Lecture 3.

- Doesn’t the Fact on page 8 in the Lecture 3 notes make much more sense now?

- Know the definition of $|x|$

- **Know the Triangle Inequality and how to prove it. More importantly, know how to use it! It’s literally one of the most important tools in this course**

- Understand the proof of the corollary to the triangle inequality (with $\text{dist}(a, b)$), it illustrates an important technique that’s used over and over again.

- Know the statement and the proof of the reverse triangle inequality (Problem 3.5)

- Know how to do AP1(a)(b) in HW 2 but ignore AP1(c)

- Ignore AP 2 in HW 2

- Also check out 3.8 in HW 2
4. Section 4: The Completeness Axiom

- This is one of the most important sections for Midterm 1 (along with sections 8 and 9)

- Define the concept of max and min and show that $S$ has a max or doesn’t have a max or a min. The examples on page 2-5 of Lecture 4 are excellent practice examples

- Know how to show (or not) that $S$ is bounded above (or below)

- Define $\sup(S)$ and $\inf(S)$ and show that $\sup(S) = M$ (or $\inf(S) = m$). The examples on page 9-11 and 13-15 of Lecture 4, as well as AP3 in HW 2 are excellent practice with that.

- Know the statement of the least upper bound property

- Know how to show that $\inf(S) = -\sup(-S)$ and to deduce the greatest lower bound property from that (page 6-7 of Lecture 5)

- Know the statement and the proof of the Archimedean property and know how to use it

- Know the statement of $\mathbb{Q}$ dense in $\mathbb{R}$. You don’t need to memorize the proof, but definitely understand it. In particular, note how the Archimedean property is used here

- Also check out Problems 4.7, 4.8, 4.14 (important), 4.16 and AP3 in HW 3. If you want more practice, also check out 4.10 and 4.15

- Ignore AP5 and AP6 in HW 3
5. Section 5: The Symbols $\infty$ and $-\infty$

This section is super short. Just know that $\sup(S) = \infty$ means $S$ is not bounded above and $\sup(S) = -\infty$ means $S$ is not bounded below. If you want more practice, check out AP3(c) from HW2 or check out 5.2 (prove those statements).

6. Section 6: A Development of $\mathbb{R}$

**IGNORE** this section, it will **NOT** be on the exam. One might even say I *cut* it out from the exam material 🙃

7. Section 7: Limits of Sequences

The only important thing in that section is the definition of a limit (Definition 7.1 or page 5 in Lecture 7) and the fact that limits are unique (pages 3-4 in Lecture 8). You don’t need to know the definition of a sequence. But check out Problem 7.4, it’s neat!

8. Section 8: A Discussion about Proofs

- This is the second important section to know for the midterm.
- Know how to do **ALL** the examples in this section and the lectures, they are all fair game and good practice with the definition of a limit. The examples in lecture include

  - Example 1: The Basics, $s_n = 3 - \frac{1}{n^2}$
  - Example 2: Simple Fraction, $s_n = \frac{2n+4}{4n+5}$
  - Example 3: Complex Fraction, $s_n = \frac{2n^3+3n}{n^3-2}$
  - Example 4: The Limit Does Not Exist, $s_n = (-1)^n$
  - Example 5: Square roots, $s_n \to s \Rightarrow \sqrt{s_n} \to \sqrt{s}$
- Example 6: \( s_n \to s \Rightarrow |s_n| \to |s| \) (see AP3 in HW 3)

**Note:** It’s important to write down **BOTH** the scratch work and the actual proof, otherwise you **WILL** lose points!

- Also know how to prove the statement about sequences that are bounded away from 0 (pages 6-9 of Lecture 8)

- Know the statement and the proof of the Squeeze Theorem (See problem 8.5)

- Of course, problems 8.1, 8.2, 8.3, 8.4, 8.7, 8.8, 8.9, and 8.10 are excellent practice problems

## 9. **SECTION 9: LIMIT LAWS FOR SEQUENCES**

- Prove limit laws such as
  
  - If \( s_n \to s \) and \( t_n \to t \), then \( s_n + t_n \to s + t \) (Page 10-11 of Lecture 8)
  
  - If \( s_n \to s \) and \( t_n \to t \) then \( s_n t_n \to st \) (Page 1-2 of Lecture 9)
  
  - If \( s_n \neq 0 \) and \( s_n \to s \neq 0 \) and \( t_n \to t \), then \( \frac{t_n}{s_n} \to \frac{t}{s} \) (Pages 2-4 of Lecture 9); you’d of course have to show the intermediate step of Example 7

- Know how to show that if \((s_n)\) converges, then \((s_n)\) is bounded (Last section in Lecture 8)

- Know how to show
  
  - Example 7: If \( s_n \neq 0 \) and \( s_n \to s \neq 0 \), then \( \frac{1}{s_n} \to \frac{1}{s} \)
- Example 8: If $|a| < 1$, then $\lim_{n \to \infty} a^n = 0$

- Example 9: $\lim_{n \to \infty} n^{\frac{1}{n}} = 1$ and its corollary $\lim_{n \to \infty} a^{\frac{1}{n}} = 1$ if $a > 0$

- Example 10: Infinite limits such as $\lim_{n \to \infty} \sqrt{n - 2} + 3 = \infty$

**Note:** Even though I haven’t explicitly done it, know how to show that $\lim_{n \to \infty} \frac{1}{n^p} = 0$ if $p > 0$ (This is just 8.1(d) but with $p$ instead of 3)

- Know the binomial theorem (but you don’t need to know how to prove it)

- Know how to define $\lim_{n \to \infty} s_n = \infty$ and $\lim_{n \to \infty} s_n = -\infty$ and know how to show that a sequence goes to $\infty$ (like Example 10 above)

- Prove some limit laws for infinite limits, such as if $s_n \to \infty$ and $t_n \to \infty$, then $s_n t_n \to \infty$ (page 11 of Lecture 9) if $s_n \to \infty$ and $t_n \geq m$ for some $m$, then $s_n + t_n \to \infty$ (Problem 9.11(c))

- Know the statement and the proof of the Duality Formula (last section of Lecture 9), and use it to show for example that $\lim_{n \to \infty} 2^n = \infty$

- Problems 9.6(a)(b), 9.9, 9.10, 9.11, 9.17, AP1, and AP2 in HW 4 are great practice problems

- Ignore 9.12 and 9.15 (they’re on HW 4 though) and ignore AP 3 on HW 4.