LECTURE 7: LIMITS OF SEQUENCES (I)

1. Sequences

Video: What is a Sequence?

Welcome to our Sequence adventure! In this chapter, we’ll study sequences, which are infinite lists of numbers.

Intuitively:

A sequence \((s_n)_{n \in \mathbb{N}}\) is an infinite list of real numbers.

Examples:

1. \(s_n = \frac{1}{n^2}, n \in \mathbb{N}\), so \((s_n) = (1, \frac{1}{4}, \frac{1}{9}, \ldots)\)

2. \(s_n = (-1)^n, n \geq 0\), so \((s_n) = (1, -1, 1, -1, 1, \ldots)\). This sequence jumps back and forth between 1 and \(-1\). It’s a great source of counterexamples.

\[
\begin{array}{c}
1 \\
\cdot \\
\cdot \\
\cdot \\
s_n = (-1)^n \\
\cdot \\
\cdot \\
\cdot \\
-1
\end{array}
\]

Date: Monday, April 13, 2020.
(3) $s_n = \cos\left(\frac{\pi n}{2}\right), n \geq 0$, which is $(1, 0, -1, 0, 1, 0, -1, 0, \cdots)$

(4) $s_n = \frac{(-1)^n}{n}$, which jumps back and forth between positive and negative values, but which seems to go to 0

**Actual Definition:**

A sequence $(s_n)_{n \in \mathbb{N}}$ is a function from $\mathbb{N}$ to $\mathbb{R}$
Why? Because for natural number $n$, you associate a real number $s_n$. For example, the sequence $s_n = \frac{1}{n^2}$ is the same as the function $f(n) = \frac{1}{n^2}$. In fact $f(1) = 1, f(2) = \frac{1}{4}, f(3) = \frac{1}{9} \cdots$.

2. LIMITS OF SEQUENCES

Video: [What is a limit?](#)

Goal: Figure out what happens to $s_n$ as $n$ goes to $\infty$.

Example: Consider $s_n = 3 - \frac{1}{n^2}$.

Intuitively, $s_n$ approaches to $s = 3$ as $n$ goes to $\infty$, and our goal is to make this rigorous.

<table>
<thead>
<tr>
<th>Intuitively:</th>
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<td>$\lim_{n \to \infty} s_n = s$ means that if $n$ is large, then $s_n$ goes to $s$</td>
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First of all, $s_n$ goes to $s$ means that $\text{dist}(s_n, s) = |s_n - s|$ is small.

In other words, we can make $|s_n - s|$ as small as we want, by letting $n$ be large enough.

That is, there is some threshold $N$ such that, after $N$, $|s_n - s|$ is as small as we want.

Finally, what does as small as we want mean? For astronomers, 10 km is small, but 10 km for microbiologists is actually pretty big! In some sense, we want a definition that makes everyone happy.

Let $\epsilon > 0$ (think error or tolerance, so $\epsilon = 10$ km for astronomers, and $\epsilon = 1$nm for biologists), then:
Ultra Important Definition of a Limit:

$$\lim_{{n \to \infty}} s_n = s$$

means:

For all $\epsilon > 0$ there is some $N$ such that if $n > N$, then $|s_n - s| < \epsilon$

That is, no matter how small our error $\epsilon$ is, there is some threshold $N$ such that, once you pass the threshold ($n > N$) then your can guarantee that $s_n$ is at most $\epsilon$ away from $s$.

Note:

$$|s_n - s| < \epsilon \Rightarrow -\epsilon < s_n - s < \epsilon \Rightarrow s - \epsilon < s_n < s + \epsilon$$

So in other words we can always guarantee that $s_n$ is in the small interval $(s - \epsilon, s + \epsilon)$
**Analogy:** Think of $s_n$ as an airplane and $(s - \epsilon, s + \epsilon)$ as a runway. What this is saying is that, no matter how small our runway is, we can always guarantee that the plane $s_n$ lands in the runway if $N$ is large enough.

For the rest of today (and the next lecture), we’ll practice with the rigorous definition of a limit, so that you can get a feel for it. This is very important for the exams.

### 3. Example 1: The Basics

**Video:** [Limit Example 1: The Basics](#)

**Example 1:**

Show

$$\lim_{n \to \infty} 3 - \frac{1}{n^2} = 3$$

Show: For all $\epsilon > 0$ there is $N > 0$ such that if $n > N$ then:

$$|s_n - s| < \epsilon$$

**STEP 1:** Find $N$

**Note:** This step is scratchwork and is technically not part of your proof. The goal here is to find $N$ and you do that by solving for $n$ in $|s_n - s| < \epsilon$:

$$|s_n - s| = \left|\left(3 - \frac{1}{n^2}\right) - 3\right| = \left|\frac{-1}{n^2}\right| = \frac{1}{n^2} < \epsilon$$
Which gives $n^2 > \frac{1}{\epsilon} \Rightarrow n > \sqrt{\frac{1}{\epsilon}} = \frac{1}{\sqrt{\epsilon}}$.

Therefore let $N = \frac{1}{\sqrt{\epsilon}}$ (Note that $N$ is not necessarily an integer)

**STEP 2:** Our actual proof:

Let $\epsilon > 0$ be given and let $N = \frac{1}{\sqrt{\epsilon}}$. Then if $n > N = \frac{1}{\sqrt{\epsilon}}$, we have:

$$|s_n - s| = \left| 3 - \frac{1}{n^2} - 3 \right| = \left| \frac{-1}{n^2} \right| = \frac{1}{n^2}$$

But if $n > \sqrt{\frac{1}{\epsilon}}$, then $n^2 > \frac{1}{\epsilon}$, so $\frac{1}{n^2} < \epsilon$, and hence

$$|s_n - s| = \frac{1}{n^2} < \epsilon$$

Therefore $\lim_{n \to \infty} 3 - \frac{1}{n^2} = 3$ \hspace{1cm} \Box

4. **Example 2: Simple Fraction**

**Video:** [Limit Example 2: Simple Fraction](#)

**Example 2:**

Show:

$$\lim_{n \to \infty} \frac{2n + 4}{4n + 5} = \frac{1}{2}$$

**Note:** Intuitively this should be true because

$$\frac{2n + 4}{4n + 5} \approx \frac{2n}{4n} = \frac{1}{2}$$

Show for all $\epsilon > 0$ there is $N$ such that if $n > N$, then $|s_n - s| < \epsilon$
**STEP 1:** Find $N$

\[ |s_n - s| = \left| \frac{2n + 4}{4n + 5} - \frac{1}{2} \right| = \left| \frac{(2n + 4)(2) - (4n + 5)}{2(4n + 5)} \right| = \left| \frac{4n + 8 - 4n - 5}{2(4n + 5)} \right| = \left| \frac{3}{2(4n + 5)} \right| > 0 \]

However,

\[ \frac{3}{2(4n + 5)} < \epsilon \]

\[ \Rightarrow \frac{1}{4n + 5} < \frac{2\epsilon}{3} \]

\[ \Rightarrow 4n + 5 > \frac{3}{2\epsilon} \]

\[ \Rightarrow 4n > \frac{3}{2\epsilon} - 5 \]

\[ \Rightarrow n > \frac{3}{8\epsilon} - \frac{5}{4} \]

This suggests to let \( N = \frac{3}{8\epsilon} - \frac{5}{4} \).
**STEP 2:** Let $\epsilon > 0$ be given, let $N = \frac{3}{8\epsilon} - \frac{5}{4}$, then if $n > N$, we have

$$|s_n - s| = \frac{3}{2(4n + 5)}$$

But if $n > N$, then

$$4n + 5 > 4 \left( \frac{3}{8\epsilon} - \frac{5}{4} \right) + 5 = \frac{3}{2\epsilon} - 5 + 5 = \frac{3}{2\epsilon}$$

Therefore $\frac{1}{4n+5} < \frac{2\epsilon}{3}$, and so

$$|s_n - s| = \frac{3}{2(4n + 5)} < \left( \frac{3}{2} \right) \left( \frac{2\epsilon}{3} \right) = \epsilon \sqrt{\epsilon}$$

Therefore $\lim_{n \to \infty} \frac{2n+4}{4n+5} = \frac{1}{2}$

**IMPORTANT:** Your absolutely **HAVE** to write down both steps, even if it seems repetitive (Because Step 1 is just scratch work to find $N$, but in step 2, you’re proving that your $N$ works). Otherwise you’ll lose points on the exam.

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5. **Example 3: A Complex Fraction**

**Video:** [Limit Example 3: A Complex Fraction](#)

**Example 3:**

Show:

$$\lim_{n \to \infty} \frac{2n^3 + 3n}{n^3 - 2} = 2$$

Intuitively this is true because $\frac{2n^3+3n}{n^3-2} \approx \frac{2n^3}{n^3} = 2$
Show for all $\epsilon > 0$ there is $N$ such that if $n > N$, then $|s_n - s| < \epsilon$.

**STEP 1:**

$$|s_n - s| = \left| \frac{2n^3 + 3n}{n^3 - 2} - 2 \right| = \left| \frac{2n^3 + 3n - 2(n^3 - 2)}{n^3 - 2} \right| = \left| \frac{3n + 4}{n^3 - 2} \right| = \frac{3n + 4}{n^3 - 2} \quad \text{if } n^3 - 2 > 0 < \epsilon$$

**Note:** $n^3 - 2 > 0 \Rightarrow n > \sqrt[3]{2}$, so we at least need $n > \sqrt[3]{2}$.

Unlike the previous problem, here the fraction is trickier. We need to analyze the numerator and denominator separately.

**Numerator:** We want $3n + 4 < $ some number. But notice that if $n > 1$, then $4n > 4$, so $4 < 4n$, so $3n + 4 < 3n + 4n = 7n$. Hence $3n + 4 < 7n$

**Denominator:** We want $n^3 - 2 > $ some large number (because we’ll take reciprocals). The idea is that, even though $n^3 - 2 < n^3$, we still have $n^3 - 2 > \frac{n^3}{2}$ for large $n$\footnote{There’s really nothing special about the factor $\frac{1}{2}$, we could have also done $\frac{n^3}{4}$, that’s completely fine} as in the picture below:
But

\[ n^3 - 2 > \frac{n^3}{2} \Rightarrow \left(1 - \frac{1}{2}\right)n^3 > 2 \Rightarrow \frac{n^3}{2} > 2 \Rightarrow n^3 > 4 \Rightarrow n > \sqrt[3]{4} \]

Hence \( n^3 - 2 > \frac{n^3}{2} \) so \( \frac{1}{n^{3-2}} < \frac{1}{n^3} \).

**Fraction:** Therefore, if both of the above conditions hold, we get:

\[
\frac{3n + 4}{n^3 - 2} < \frac{7n}{n^3} = \frac{14n}{n^3} = \frac{14}{n^2}
\]

And therefore

\[
\frac{14}{n^2} < \epsilon \Rightarrow \frac{n^2}{14} > \frac{1}{\epsilon} \Rightarrow n^2 > \frac{14}{\epsilon} \Rightarrow n > \sqrt{\frac{14}{\epsilon}}
\]
This suggests to let $N = \sqrt{\frac{14}{\epsilon}}$, but since we also need $n > \sqrt[3]{2}$, $n > 1$ and $n > \sqrt[3]{4}$ (see boxed numbers above), $N$ actually needs to be the larger one of those 4 numbers, in other words $N$ is the $\text{max}$ of $\sqrt[3]{2}, 1, \sqrt[3]{4}, \sqrt{\frac{14}{\epsilon}}$

**STEP 2:** Let $\epsilon > 0$ and let $N = \max \left\{ \sqrt[3]{2}, 1, \sqrt[3]{4}, \sqrt{\frac{14}{\epsilon}} \right\} = \max \left\{ \sqrt[3]{4}, \sqrt{\frac{14}{\epsilon}} \right\}$ (since $\sqrt[3]{4} > 1$ and $\sqrt[3]{4} > \sqrt[3]{2}$)

Then if $n > N$, we have:
\[|s_n - s| = \left| \frac{2n^3 + 3n}{n^3 - 2} - 2 \right| \]
\[= \left| \frac{3n + 4}{n^3 - 2} \right| \]
\[= \frac{3n + 4}{n^3 - 2} \quad \text{Since } n > \sqrt[3]{2} \text{, so } n^3 - 2 > 0 \]
\[< \frac{7n}{n^3 - 2} \quad \text{Since } n > 1 \text{ so } 3n + 4 < 3n + 4n = 7n \]
\[= \frac{7n}{n^3 - 2} \quad \text{Since } n > \sqrt[3]{4} \text{ so } n^3 - 6 > \frac{n^3}{2} \]
\[= \frac{14}{n^2} \]

But
\[n > \sqrt{\frac{14}{\epsilon}} \Rightarrow n^2 > \frac{14}{\epsilon} \Rightarrow \frac{1}{n^2} < \frac{\epsilon}{14} \]

Therefore:
\[|s_n - s| = \frac{14}{n^2} < 14 \left( \frac{\epsilon}{14} \right) = \epsilon \sqrt{} \]

Hence \(\lim_{n \to \infty} \frac{2n^3 + 3n}{n^3 - 2} = 2\)