Today: Let’s conclude our real number adventure with two little topics related to real numbers: $\infty$ and the construction of $\mathbb{R}$.

1. What is $\infty$?

Video: [What is Infinity?](#)

**Definition:**
$\infty$ is a symbol that is larger than any real number. In other words, for all $x \in \mathbb{R}$, $x < \infty$.

**Important:** $\infty$ is **not** a real number! In particular, it does not follow the same algebraic laws that $\mathbb{R}$ does. For instance, even though for all $x \in \mathbb{R}$, $x - x = 0$, we do **NOT** have $\infty - \infty = 0$ (see this video if you’re curious why).

The nice thing is that now we can extend the definition of $\text{sup}(S)$ and $\text{inf}(S)$ in the case where $S$ is unbounded:

**Definition:**
We say $\text{sup}(S) = \infty$ if $S$ is not bounded above, that is: for all $M$ there is $s_1 \in S$ such that $s_1 > M$.  

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*Date: Friday, April 10, 2020.*
Example: If $S = (3, \infty)$ then $\text{sup}(S) = \infty$

**Definition:**

We say $\text{inf}(S) = -\infty$ if $S$ is not bounded below, that is: for all $m$ there is $s_1 \in S$ such that $s_1 < m$.

Example: If $S = (-\infty, 1)$ then $\text{inf}(S) = -\infty$

With this new definition, the least upper bound property is quite elegant:

**Least Upper Bound Property:**

If $S$ is a nonempty subset of $\mathbb{R}$, then either $\text{sup}(S)$ exists or is $\infty$ (similarly, either $\text{inf}(S)$ exists or is $-\infty$).

(Why? Either $S$ is bounded above and $\text{sup}(S)$ exists, or $S$ is not bounded above and $\text{sup}(S) = \infty$).
Moreover, most of the statements that we’ve shown so far, like $\inf(S) = -\sup(-S)$ also hold for the case where $\sup(S) = \infty$ or $\inf(S) = -\infty$.

**Example:**

Find $\sup(S)$ where

$$S = \left\{ n^2(-1)^n \mid n \in \mathbb{N} \right\} = \{-1, 4, -9, 16, \cdots \}$$

**Picture:** Notice $n^2(-1)^n$ jumps back and forth between $y = x^2$ and $y = -x^2$.

Let $M$ be given, we need to find some $s_1 = n^2(-1)^n \in S$ such that $s_1 > M$. 
Scratchwork: Notice that \( n^2(-1)^n \) gets bigger for even \( n \). Moreover, if \( n \) is even, then \( n^2(-1)^n = n^2 \). Also \( n^2 > M \Rightarrow n > \sqrt{M} \).

Actual Proof: Let \( n \) be any even integer such that \( n > \sqrt{|M|} \) and let \( s_1 = n^2(-1)^n \in S \); then

\[
s_1 = n^2(-1)^n = n^2 > (\sqrt{M})^2 = M
\]

Hence \( \sup(S) = \infty \).

2. Construction of \( \mathbb{R} \)

Video: Construction of \( \mathbb{R} \)

\[^1\]In the case \( M < 0 \), just let \( n = 2 \)
Heads-up: This section is tricky, but you’ll only need it for the homework and the quizzes; I won’t ask about it on the exams. It’s incredibly fascinating though!

Even though we’ve been talking for weeks about the real numbers, we never actually defined what a real number is! That’s precisely what we’re going to do now. As an added benefit, we’ll be able to prove the least upper bound property (so it’s not an axiom after all!)

Goal: Construct the real numbers from the rational numbers.

Motivation: How would you define $\sqrt{2}$ using only rational numbers, without ever mentioning the number $\sqrt{2}$?

Consider the following set $S$ (recall $\sqrt{2} \approx 1.414$)

$$S = \left\{ r \in \mathbb{Q} \mid r < \sqrt{2} \right\} = \left\{ 1, -2, \frac{4}{7}, 0, 1.2, 1.41, -3.6, \cdots \right\}$$

In other words, $S$ is the set of all rational numbers that come before $\sqrt{2}$.

Upshot: Strictly speaking, the set $\left\{ 1, -2, \frac{4}{7}, 0, 1.2, 1.41, -3.6, \cdots \right\}$ doesn’t mention $\sqrt{2}$ at all; to the naked eye, it is just a random set of rational numbers. And that’s precisely how we’ll define real numbers,
simply as *special* sets of rational numbers.

**Definition:**

A real number $S$ is a subset of $\mathbb{Q}$ with the following properties (called a Cut):

1. $S \neq \emptyset$ and $S \neq \mathbb{Q}$
2. If $r \in S$ and if $s$ is any rational with $s < r$, then $s \in S$
3. If $r \in S$, then there is some $s \in S$ such that $s > r$

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**Note:** (2) is just saying that $S$ contains all the rationals before $r$, like in the following picture:

![Diagram](image)

**Note:** (3) is saying that $S$ has no maximum: No matter which $s \in S$ you pick, there's always a bigger element $r \in S$
Notice the subtle difference between (2) and (3). In (2) we say for all $s < r$, $s \in S$, but in (3) we say there is $s \in S$ with $s > r$

**Important Example:** If $a \in \mathbb{Q}$ is given, then

$$S = a^* = \{ r \in \mathbb{Q} \mid r < a \}$$

This is a rational cut and written as $a^*$.

So for instance

$$1^* = \{ r \in \mathbb{Q} \mid r < 1 \} = \{-2.3, -1, 0, 0.5, 0.99, \cdots \}$$
Non-Examples:

(a) $\emptyset, \mathbb{Q}$: doesn’t satisfy (1)

(b) $S = \{ r \in \mathbb{Q} \mid 0 < r < 2 \}$: Doesn’t satisfy (2): $1 \in S$ and $-1 < 1$ but $-1 \notin S$. Similarly, $S = \{ r \in \mathbb{Q} \mid r > 2 \}$ is not a cut: $3 \in S$ and $-1 < 3$ but $-1 \notin S$

(c) $S = \{ r \in \mathbb{Q} \mid r \leq \frac{1}{2} \}$: Doesn’t satisfy (3) because $S$ has a maximum of $\frac{1}{2}$: $r = \frac{1}{2} \in S$ but there is no $s \in S$ with $s > \frac{1}{2}$
Examples:

(a) $S = \{1, -2, \frac{4}{7}, 0, 1.2, 1.41, -3.6, \cdots \}$ (again some specific set of rational numbers, all that are $< \sqrt{2}$) is a cut, called $\sqrt{2}$ (but see a more concrete version below).

(b) $S = \{r \in \mathbb{Q} \mid r^3 < 2\}$ is a cut, called $\sqrt[3]{2}$.

(d) $S = \{r \in \mathbb{Q} \mid r^2 < 2\}$ is NOT a cut: $0 \in S$ and $-4 < 0$ but $-4 \notin S$. BUT $S = \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{r \in \mathbb{Q} \mid r^2 < 2\}$ is a cut, called $\sqrt{2}$.

Now that we defined what a cut is, let’s see what operations we can do on them (just like operations on real numbers)
**Definition:**
If $S$ and $T$ are cuts, then

$$S + T = \{ s + t \mid s \in S \text{ and } t \in T \}$$

**Fact:**
If $S$ and $T$ are cuts, then $S + T$ is a cut

**Note:** Multiplication of cuts is trickier to define; in particular $S \cdot T$ is NOT $\{ st \mid s \in S \text{ and } t \in T \}^2$

**Definition:**
$$\mathbb{R} = \text{Set of all cuts in } \mathbb{Q}$$

**Fact:**
With $+$ and $\cdot$ defined at above, $\mathbb{R}$ is a field (section 3)

Lastly, we can define an ordering on cuts simply as follows:

**Definition:**
If $S$ and $T$ are cuts, then $S \leq T$ means $S \subseteq T$

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$^2$See Pugh’s book if you want to see how to define $S \times T$
For example:

\[ 1^* = \{ r \in \mathbb{Q} \mid r < 1 \} \subseteq \{ r \in \mathbb{Q} \mid r < 2 \} = 2^* \]

So by definition \( 1^* \leq 2^* \)

**Fact:**

With \( \leq \) defined at above, \( \mathbb{R} \) becomes an ordered field (section 3)

Since we can identify rational numbers \( a \) with rational cuts \( a^* \), with this identification, we can say that \( \mathbb{R} \) “includes” \( \mathbb{Q} \) (although, strictly speaking, elements of \( \mathbb{R} \) are cuts, but elements of \( \mathbb{Q} \) are rational numbers)
3. Proof of the Least Upper Bound Property

Video: Least Upper Bound Property Proof

Why care about cuts? Because using them we can easily prove the least upper bound property. It’s the elegance of the proof below that is the fruit of all our hard labor!

Least Upper Bound Property for Cuts:

If $S$ is a nonempty set of cuts (= real numbers) that is bounded above, then $S$ has a least upper bound.

Proof: Let $M$ be the union of all the cuts $S \in S$, that is:

$$M = \{ r \in \mathbb{Q} \mid r \in S \text{ for some } S \in S \}$$
(STEP 1) **Claim:** $M$ is a cut

First of all, $M \neq \emptyset$ because any cut $S$ is nonempty (by definition) and $M$ is just the union over all the cuts $S$.

Let’s show $M \neq \mathbb{Q}$. Let $B$ be an upper bound for $S$ (which exists by our assumption). By definition of upper bound, for all $S \in \mathcal{S}$, $S \leq B$, meaning that $S \subseteq B$. Therefore, if you take the union $\cup S$ over all $S \in \mathcal{S}$, it is still true that $M = \cup S \subseteq B$, hence $M \subseteq B$ and since $B \neq \mathbb{Q}$ (Because $B < \infty$) we obtain $M \neq \mathbb{Q}$.
Suppose $r \in M$ and $s < r$. By definition of $M$, there is $S \in \mathcal{S}$ with $r \in S$. But since $s < r$ and $S$ is a cut, we get $s \in S \subseteq M$, so $s \in M$. ✓

Suppose $r \in M$. By definition of $M$, there is $S \in \mathcal{S}$ with $r \in S$. But then since $S$ is a cut, there is $s \in S$ with $s > r$. Since $S \subseteq M$, we get $s \in M$. So there is $s \in M$ such that $s > r$. ✓

(STEP 2) **Claim:** $M$ is an upper bound for $\mathcal{S}$.

This just follows from the definition of $M$ as a union: Namely if $S \in \mathcal{S}$, then by definition of $M$, $S \subseteq M$. So for all $S \in \mathcal{S}$, $S \leq M$. ✓
(STEP 3) **Claim:** $M$ is the least upper bound for $S$

Let $M_1$ be any other upper bound for $S$, meaning for all $S \in S$, $S \leq M_1$, that is $S \subseteq M_1$. Then, if you take the union over all $S \in S$, we get $\cup S \subseteq M_1$, that is $M \subseteq M_1$ (by definition of $M$), so $M \leq M_1$. This means that $M$ is indeed the *least* upper bound: any other upper bound $M_1$ must be greater than or equal to $M$. $\square$