LECTURE 3: THE SET $\mathbb{R}$ OF REAL NUMBERS

1. **What is a field?**

*Video: [What is a field?]*

Let’s get real and talk about real numbers! Just like we did for $\mathbb{N}$, we will again define $\mathbb{R}$ in terms of axioms. In other words, what properties make $\mathbb{R}$ so special?

First of all, what distinguishes $\mathbb{R}$ from $\mathbb{N}$ or $\mathbb{Z}$ is that $\mathbb{R}$ is a **field**: 

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**Definition:**

A field $\mathbb{F}$ is a set equipped with two operations: addition $+$ and multiplication $\cdot$ such that the following properties are true. Here $a, b, c \in \mathbb{F}$

**Addition Axioms:**

(A0) $a, b \in \mathbb{F} \Rightarrow a + b \in \mathbb{F}$ (closed under $+$)

(A1) $(a + b) + c = a + (b + c)$ (associativity)

(A2) $a + b = b + a$ (commutativity)

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*That is functions from $\mathbb{F} \times \mathbb{F}$ to $\mathbb{F}$*

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*Date: Friday, April 3, 2020.*
(A3) There is an element $0 \in \mathbb{F}$ such that $a + 0 = 0 + a = a$ for all $a \in \mathbb{F}$ (zero-element)

(A4) For all $a \in \mathbb{F}$ there is an element $-a \in \mathbb{F}$ such that $a + (-a) = (-a) + a = 0$ (additive inverse)

**Multiplication Axioms:**

(M0) $a, b \in \mathbb{F} \Rightarrow ab \in \mathbb{F}$ (closed under $\cdot$)

(M1) $(ab)c = a(bc)$ (associativity)

(M2) $ab = ba$ (commutativity)

(M3) There is an element $1 \in \mathbb{F}$ such that $1a = a1 = a$ for all $a \in \mathbb{F}$ (1-element)

(M4) For all $a \neq 0$ there exists an element $a^{-1} \in \mathbb{F}$ such that $aa^{-1} = a^{-1}a = 1$

**Distributive law:**

(DL) $a(b + c) = ab + ac$, $(a + b)c = ac + bc$

**Examples:** $\mathbb{R}$ (of course), but also $\mathbb{Q}$ (rational numbers), $\mathbb{C}$ (complex numbers) and even $\{0, 1\}$ (with addition defined as $1 + 1 = 0$).

**Non-Examples:** $\mathbb{N}$ (generally if $n \in \mathbb{N}$, $-n \notin \mathbb{N}$), $\mathbb{Z}$ (generally if $m \in \mathbb{Z}$, then $m^{-1} \notin \mathbb{Z}$), so already this distinguishes $\mathbb{R}$ from $\mathbb{N}$ and $\mathbb{Z}$

Again, hopefully those axioms are “obvious” to you; they are meant to be a good model of $\mathbb{R}$, and in fact a lot of properties of $\mathbb{R}$ are true for general fields. Here are some natural consequences of our field axioms:
Theorem:
The following properties are true in any field $\mathbb{F}$. Here $a, b, c$ are arbitrary elements in $\mathbb{F}$:

1. $a + c = b + c \Rightarrow a = b$ (cancellation law)
2. If $a \neq 0$, then $ab = ac \Rightarrow b = c$ (another cancellation law)
3. $a0 = 0$
4. $(−a)b = −ab$
5. $(−a)(−b) = ab$
6. $ab = 0 \Rightarrow a = 0$ or $b = 0$ ($\mathbb{F}$ is an integral domain)

Proof:
(1)

\[
a + c = b + c \\
\Rightarrow (a + c) + (−c) = (b + c) + (−c) \\
\Rightarrow a + (c + (−c)) = b + (c + (−c)) \quad \text{(Associativity)} \\
\Rightarrow a + 0 = b + 0 \quad \text{(Definition of } −c) \\
\Rightarrow a = b \quad \text{(Definition of 0)}
\]

(2) Similar (Multiply by $a^{-1}$)

(3) First of all, by definition of 0, we have $0 + 0 = 0$ (you’re adding nothing to 0). Now consider $0 + a0$: 
0 + a0 \\
= a0 (Definition of 0) \\
= a(0 + 0) \\
= a0 + a0 (Distributivity)

Hence 0 + a0 = a0 + a0, and canceling out a0 (by (1)) we get 0 = a0, so a0 = 0

(4) Consider

\[ ab + (-a)b = (a + (-a))b \] (Distributivity) \\
= 0b (Definition of \(-a\)) \\
= 0 (By (3))

Hence \( ab + (-a)b = 0 \), so \((-a)b\) is the additive inverse of \(ab\), that is, \((-a)b = -ab\) (by definition of \(-ab\))

(5) Skip (basically apply (4) twice)

(6) Suppose \(ab = 0\) but \(a \neq 0\), then

\[ ab = 0 \] \\
\[ a^{-1}(ab) = a^{-1}(0) \] \\
\[ (a^{-1}a)b = 0 \] (Associativity and (1)) \\
\[ 1b = 0 \] (Definition of \(a^{-1}\)) \\
\[ b = 0 \] (Definition of 1)
Note: If you’re interested in learning more about fields, make sure to take Math 120B (which is all about fields)

2. Ordered Fields

Video: [Ordered Fields](#)

That said, there is more to \( \mathbb{R} \) than just being a field. In particular, notice that in \( \mathbb{R} \) we can compare elements, like saying \( 2 \leq 3 \). This distinguishes \( \mathbb{R} \) from its parent \( \mathbb{C} \) (we cannot compare complex numbers, see homework or [this video](#)).

**Definition:**

A field \( \mathbb{F} \) is called an **ordered field** if it has a structure \( \leq \) that satisfies the following. Here \( a, b, c \in \mathbb{F} \):

- \((O1)\) Either \( a \leq b \) or \( b \leq a \) (Trichotomy)
- \((O2)\) \( a \leq b \) and \( b \leq a \) \( \Rightarrow \) \( a = b \) (also Trichotomy)
- \((O3)\) \( a \leq b \) and \( b \leq c \) \( \Rightarrow \) \( a \leq c \) (Transitivity)
- \((O4)\) \( a \leq b \) \( \Rightarrow \) \( a + c \leq b + c \) (Addition preserves order)
- \((O5)\) \( a \leq b \) and \( 0 \leq c \) \( \Rightarrow \) \( ac \leq bc \) (Nonnegative multiplication preserves order)

\( \leq \) is a function from \( \mathbb{F} \times \mathbb{F} \) to \{ True, False \}

**Note:** \( a \geq b \) is defined to be \( b \leq a \) and \( a < b \) means “\( a \leq b \) and \( a \neq b \)” (Similar for \( a > b \))
Examples: \( \mathbb{R} \) and \( \mathbb{Q} \)

Non-Examples: \( \mathbb{Z} \) (not a field), \( \mathbb{C} \) (cannot order elements)

And of course, from those axioms one can prove other neat facts:

**Theorem:**

The following properties are true in any ordered field \( \mathbb{F} \). Here \( a, b, c \) are arbitrary elements in \( \mathbb{F} \):

1. \( a \leq b \Rightarrow -a \geq -b \)
2. \( a \leq b \) and \( c \leq 0 \Rightarrow ac \geq bc \)
3. \( b \geq 0 \) and \( c \geq 0 \Rightarrow bc \geq 0 \)
4. \( a^2 \geq 0 \)
5. \( 0 < 1 \)
6. \( a > 0 \Rightarrow a^{-1} > 0 \)
7. \( a > b > 0 \Rightarrow a^{-1} < b^{-1} \)

**Proof:**

1. 
   \[
   a \leq b \Rightarrow a + ((-a) + (-b)) \leq b + ((-a) + (-b)) \quad \text{(Since + preserves order)}
   \]
   \[
   -b \leq -a
   \]
   \[
   -a \geq -b \quad \text{(By definition of \( \geq \))}
   \]

2. First note that if \( c \leq 0 \), then \(-c \geq 0\) (by (1)), but then, by \((O5)\),
\[ a \leq b \\
(-c)a \leq (-c)b \\
-ac \leq -bc \\
ac \geq bc \text{ (By (1))} \]

(3) This is just \((O5)\) with \(a = 0\)

(4) By trichotomy, we know either \(a \geq 0\) or \(a \leq 0\).

Case 1: If \(a \geq 0\), then by \((O5)\), we get \(aa \geq a0\), so \(a^2 \geq 0\) ✓

Case 2: If \(a \leq 0\), then by (3), we get \(aa \geq a0\), so \(a^2 \geq 0\) ✓

So in any case \(a^2 \geq 0\)

(5) Follows from (4) because \(1 = 1^2 \geq 0\), so to conclude, all that’s left to show is that \(1 \neq 0\) (skip)

(6) Suppose \(a > 0\) but \(a^{-1} \leq 0\). Since \(a \geq 0\), we have \(aa^{-1} \leq a0\), so \(1 \leq 0\), but this contradicts (5) ⇒⇐

(7) (Skip; Start with \(a < b\) and multiply by \(b^{-1}\) and then by \(a^{-1}\))

Now you may have noticed that everything above is not just valid for \(\mathbb{R}\), but also for \(\mathbb{Q}\). In particular being an ordered field isn’t what makes \(\mathbb{R}\) so special.

But then what makes \(\mathbb{R}\) so special? Unfortunately we won’t be able to answer that question until section 4, but let me already tell you the answer.
**Note:** A subfield $B$ is a subset $A \subseteq B$ that is also a field. For example, $\mathbb{Q}$ is a subfield of $\mathbb{R}$ since $\mathbb{Q} \subseteq \mathbb{R}$ but $\mathbb{Q}$ is a field (kind of like a subspace of a vector space in 121A)

**Fact:**

There exists an ordered field called $\mathbb{R}$ that contains $\mathbb{Q}$ as a subfield and which satisfies the least upper bound property (see section 4)

**Note:** In this course we’ll take $\mathbb{R}$ as a given, but I’d like to point out that there is an explicit construction of $\mathbb{R}$ in section 6 (which we’ll briefly go over)

Again, the least upper bound property will be discussed in section 4, but intuitively it is saying that, unlike $\mathbb{Q}$, $\mathbb{R}$ has no holes, as in the following picture:
In some sense, $\mathbb{Q}$ has lots of gaps, but $\mathbb{R}$ fills those gaps, that’s why $\mathbb{R}$ is so much nicer than $\mathbb{Q}$.

### 3. Triangle Inequality

**Video:** [Triangle Inequality](#)

Last but not least, I would like to remind you of the most important inequality in this course: the triangle inequality. For this, let’s recall the concept of absolute value from Math 2A:

**Definition:**

$$|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x \leq 0 
\end{cases}$$
From this one can show things like \(|x| \geq 0\) for all \(x\) (use the definition) and \(|ab| = |a||b|\) (do it by cases, for example \(a \geq 0\) and \(b \geq 0\), see book), and most importantly:

**Triangle Inequality:**

\[|a + b| \leq |a| + |b|\]

**Proof:**

**STEP 1:** We first need a small lemma:

**Lemma:**

\[-|x| \leq x \leq |x|\]

**Proof of Lemma:**

**Case 1:** \(x \geq 0\), then \(|x| = x\) so \(x = |x| \leq |x|\), and moreover \(x \geq 0 \geq -|x| \checkmark\)

**Case 2:** \(x \leq 0\), then \(|x| = -x\), so \(x = -|x| \geq -|x|\) and moreover \(x \leq 0 \leq |x| \checkmark\)

**STEP 2:** By the Lemma, we have \(-|a| \leq a \leq |a|\).

Now, add \(b\) to both sides of \(a \leq |a|\) to get \(a + b \leq |a| + b \leq |a| + |b|\)

On the other hand, add \(b\) to both sides of \(a \geq -|a|\) to get \(a + b \geq -|a| + b \geq -|a| - |b| = -(|a| + |b|)

Therefore we have:

\[-(|a| + |b|) \leq a + b \leq |a| + |b|\]

**STEP 3:** Finally, to prove \(|a + b| \leq |a| + |b|\), we do it by cases:
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Case 1: If $a + b \geq 0$, then $|a + b| = a + b \leq |a| + |b|$ (by STEP 2) √

Case 2: If $a + b \leq 0$, then $|a + b| = -(a + b) \leq -(|a| + |b|)$ (by STEP 2) = $|a| + |b|$. √

So in both cases we have the desired result □

Why is it called the triangle inequality? This will be clearer after the next result

**Definition:**

$$dist(a, b) = |a - b|$$

**Note:** This is sometimes written as $d(a, b)$
Corollary:
\[ \text{dist}(a, c) \leq \text{dist}(a, b) + \text{dist}(b, c) \]

Proof:

\[
\begin{align*}
\text{dist}(a, c) &= |a - c| \\
&= |a - b + b - c| \\
&= |(a - b) + (b - c)| \\
&\leq |a - b| + |b - c| \\
&= \text{dist}(a, b) + \text{dist}(b, c)
\end{align*}
\]

Important Note: This trick with adding/subtracting \( b \) is \textbf{SUPER} important and will be used many times over!!!

\textbf{Note:} This corollary explains why the triangle inequality is called as such. It says that the sum of the lengths of two legs of a triangle is always greater than or equal to the length of the third one. In this picture, the green segment is smaller than the sum of the red and blue ones:
A less useful inequality to note is the

**Reverse Triangle Inequality:**

\[ |a - b| \geq ||a| - |b|| \]

**Example:**

\[ |3 - (-5)| \geq ||3| - |-5|| = |3 - 5| = 2 \]

**Proof:** See HW

**Note:** The reverse triangle inequality *sounds* useful but is actually really useless, it rarely works.