LECTURE 13: SUBSEQUENCES (I)

1. INTRODUCTION

Video: [What is a Subsequence?]

Today we’ll define the concept of a subsequence, which is an analog of an express train, but for sequences.

Suppose we have a sequence \((s_n)\). Think of \((s_n)\) as a train that goes through cities (such as Peyamgeles, Liouville, Sup Francisco, or Indianapolis).

Then a subsequence \((s_{n_k})\) is an express train, that is the subsequence goes through the same cities as \((s_n)\), but doesn’t stop at every city.

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In the picture above, $s_{n_1}$ (the first express stop) is the second city, $s_{n_2}$ is the 3rd city, $s_{n_3}$ is the 5th city, and $s_{n_4}$ is the 8th city.

**Definition:**

A subsequence of $(s_n)$ is a sequence of the form $(s_{n_k})$ (with $k \in \mathbb{N}$) where for every $k$, you associate a value $s_{n_k}$ of $(s_n)$. Moreover, $n_1 < n_2 < n_3 < \ldots$.

The first condition just says that every express stop $s_{n_k}$ has to be one of the original stops of $(s_n)$, but $s_{n_k}$ can skip some of the cities.

The second condition says that the second stop $s_{n_2}$ comes after the first stop $s_{n_1}$, and so on. In other words, the express train is going forwards, not backwards (this will be important later on).

**Example 1:**

Let $(s_n)$ be the sequence:

$$s_n = (-1)^n n^2 = (-1, 4, -9, 16, -25, \ldots)$$

Define the subsequence $(s_{n_k})$ by $s_{n_k} = k^{th}$ positive term of $(s_n)$

That is, just look at the positive terms of $(s_n)$ and skip the negative ones. Or, in other words, just look at every second term of $s_n$. 
Therefore:

\[ s_{n_1} = s_2 = 4 \]
\[ s_{n_2} = s_4 = 16 \]
\[ s_{n_3} = s_6 = 36 \]

And, following that pattern, you might guess that:

\[ s_{n_k} = s_{2k} = (2k)^2 = 4k^2 \]

**Remark:** Notice that, if you define \( \sigma(k) = 2k \), then we have:

\[ s_{n_k} = s_{2k} = s_{\sigma(k)} \]

In other words, \( \sigma(k) \) just takes \( k \) and gives us \( n = 2k \), and our sequence \( s_n \) takes \( n = 2k \) as an input and spits out \( s_{2k} \). And a subsequence is nothing else than the composition of the two.
Example 2:

Let \((s_n)\) be the sequence:

\[ s_n = \sin \left( \frac{\pi n}{2} \right) = (1, 0, -1, 0, 1, 0, -1, 0, \ldots) \]
One example of a subsequence is

\((s_{n_k}) = (1, -1, 0, 0, 0, \ldots)\)

Notice that, even though the original sequence \((s_n)\) doesn’t converge, the subsequence \((s_{n_k})\) converges! And we’ll see next time that any (bounded) sequence will always have a convergent subsequence!

But if \((s_n)\) converges to \(s\), then any subsequence must also converge to \(s\). This makes sense: If a train leads you to a final destination, then any express train must also go to that final destination.

**Fact:**

If \(\lim_{n \to \infty} s_n = s\), then \(\lim_{k \to \infty} s_{n_k} = s\) as well.

**Note:** The proof of this relies on the following little fact:\(^1\)

**Mini-Fact:**

For all \(k, n_k \geq k\)

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\(^1\)The proof is by induction: The inductive step is \(n_{k+1} > n_k \geq k\), so \(n_{k+1} > k\) and therefore \(n_{k+1} \geq k + 1\) since \(k\) is an integer. For example, if \(n_{k+1} > 6\) then \(n_{k+1} \geq 7\)
In other words, the $k$th express stop must be after the $k$th non-express stop. Otherwise, why would it be called an *express* train if it’s not faster than a regular train?

**Proof of Fact:**

**Idea of Proof:** If the regular train is already close to your destination, then the express train is even closer to your destination.

Let $\epsilon > 0$ be given, since $(s_n)$ converges to $s$, there is $N$ such that for all $n > N$, then

$$|s_n - s| < \epsilon$$

Now for the same $N$, if $k > N$, then $n_k \geq k > N$, so $k > N$ and therefore

$$|s_{n_k} - s| < \epsilon$$
2. **Inductive Construction**

Video: [Link to Inductive Construction Video]

Now that we got the basics down, let’s cover an important technique called an inductive construction (of subsequence). This technique is used over and over again in analysis.

**Example 3:**

Consider the following sequence $(r_n)$ (see picture)

```
START  r_1  r_2
-3      -2  \leftarrow -1  \rightarrow 0  \rightarrow 1  \rightarrow 2  \rightarrow 3
\uparrow  \downarrow  \uparrow
-3/2    -2/2  \leftarrow -1/2  \leftarrow 0  \leftarrow 1/2  \leftarrow 2/2  \leftarrow 3/2
\uparrow  \downarrow
-3/3    -2/3  \rightarrow -1/3  \rightarrow 0  \rightarrow 1/3  \rightarrow 2/3  \rightarrow 3/3
\uparrow  \leftarrow  \leftarrow  \leftarrow  \leftarrow  \downarrow
```

Here the first row is all integers, the second row all fractions with denominator 2, the third row all fractions with denominator 3 etc.
This sequence was used in Math 13 to show that $\mathbb{Q}$ is countable! (check out the beginning of this video for a refresher: $\mathbb{R}$ is uncountable)

**Notice:** $(r_n)$ really goes through all the rational numbers, that is every rational number $r$ is of the form $r_n$ for some $n$.

But in fact, even more can be said!

**Fact:**

For any real number $a$, there is a subsequence $(r_{n_k})$ of $(r_n)$ that converges to $a$

In other words, you can not only visit every rational city using the train $(r_n)$, but you can even visit every real city if you use express trains and limits!

This is unbelievable! Because even though $(r_n)$ is pretty chaotic and doesn’t converge, for any $a$ you can find a converging subsequence that converges to $a$. In other words, there is order in chaos.
**Note:** The fact above really makes sense *because* \( \mathbb{Q} \) is dense, so given any real number \( a \), you can find arbitrarily close rational numbers \( r \). But if \( \mathbb{Q} \) were not dense, there would be a real number \( a \) that you cannot approximate with rational numbers!

**Proof:**

**Goal:** Construct a subsequence \((r_{n_k})\) with the property that for every \( k \in \mathbb{N} \)

\[
|r_{n_k} - a| < \frac{1}{k}
\]
Idea: First construct \( r_{n_1} \) and then, given \( r_{n_k} \), construct \( r_{n_{k+1}} \) (just like induction)

**STEP 1:** (Kind of like a base case)

Construct \( r_{n_1} \)

**Note:** We want

\[
|r_{n_1} - a| < \frac{1}{1} = 1
\]

\[\Rightarrow |r_{n_1} - a| < 1\]

\[\Rightarrow -1 < r_{n_1} - a < 1\]

\[\Rightarrow a - 1 < r_{n_1} < a + 1\]

Consider the interval \((a - 1, a + 1)\)
Since \( \mathbb{Q} \) is dense in \( \mathbb{R} \), there is a rational number \( r \) with \( a-1 < r < a+1 \). Now since every rational number \( r \) is of the form \( r = r_n \) for some \( n \) (remember that the train \( (r_n) \) goes through every rational city), let choose \( r_{n_1} \) to be that \( r \).

**STEP 2:** (Kind of like an inductive step)

Suppose you found \( r_{n_1}, r_{n_2}, \ldots, r_{n_k} \) with \( n_1 < n_2 < \cdots < n_k \) and

\[
|r_{n_1} - a| < 1, |r_{n_2} - a| < \frac{1}{2}, \ldots, |r_{n_k} - a| < \frac{1}{k}
\]

**Goal:** Find \( r_{n_{k+1}} \) with \( r_{n_1} < r_{n_2} < \cdots < r_{n_k} < r_{n_{k+1}} \) and

\[
|r_{n_{k+1}} - a| < \frac{1}{k+1}
\]

**Note:** It is not quite enough to apply the density argument above, since this doesn’t guarantee that \( n_{k+1} > n_k \).
**Example:** Suppose $n_k = 20$. The above density argument guarantees that $r = r_{n_{k+1}}$ for some $n_{k+1}$, but then $n_{k+1}$ could be 19, which we don’t want!

To get around that, notice that in the interval $(a - \frac{1}{k+1}, a + \frac{1}{k+1})$, there’s not only 1 rational number $r$, but in fact *infinitely* many of them!

In particular, there must definitely be a rational number $r$ that is different from $r_1, r_2, \ldots, r_{n_k}$ (which is just a finite number of rationals). Choose $r_{n_{k+1}}$ to be that rational number.

Then since $r_{n_{k+1}}$ is in $(a - \frac{1}{k+1}, a + \frac{1}{k+1})$ and also $n_{k+1} > n_k$ since $r_{n_{k+1}}$ is different from the first $n_k$ rational numbers $r_1, r_2, \ldots, r_{n_k}$ ✓

(Analogy: $r_{n_{k+1}}$ is a brand new city, so it must be stop you haven’t seen before. It lies in the future rather than the past)
STEP 3: Therefore, by this inductive construction, we have found a subsequence \((r_{nk})\) such that \(|r_{nk} - a| < \frac{1}{k}\) for every \(k\), and lastly, because \(\lim_{k \to \infty} \frac{1}{k} = 0\), by the squeeze theorem, \(\lim_{k \to \infty} r_{nk} = a\) \(\square\)

3. ANOTHER INDUCTIVE CONSTRUCTION

Video: [Another Inductive Construction](#)

Here is yet another enlightening example of an inductive construction:

**Example 4:**
Suppose \((s_n)\) is a sequence with \(s_n > 0\) for all \(n\) and \(\inf \{s_n \mid n \in \mathbb{N}\} = 0\). Show that \((s_n)\) has a decreasing subsequence \((s_{nk})\) converging to 0.
This result is really surprising! \((s_n)\) not only has a subsequence going to 0, but that one is actually decreasing!

**Goal:** Construct a subsequence \((s_{n_k})\) with \(s_{n_{k+1}} < s_{n_k}\) for all \(k\) and

\[
s_{n_k} < \frac{1}{k}.
\]

**STEP 1:** Want \(s_{n_1} < \frac{1}{1} = 1\).

But notice that:

\[
1 > 0 = \inf \{s_n\}
\]

So by definition of the inf (Analogy: You’re not the worst student), there is \(s_{n_1}\) with \(s_{n_1} < 1\) (There’s someone worse than you). ✓

**STEP 2:** Suppose we found \(s_{n_1}, s_{n_2}, \ldots, s_{n_k}\) such that \(s_{n_1} > s_{n_2} > \cdots > s_{n_k}\) and \(s_{n_1} < 1, s_{n_2} < \frac{1}{2}, \ldots, s_{n_k} < \frac{1}{k}\).

We’d like to find \(s_{n_{k+1}}\) with \(s_{n_{k+1}} < s_{n_k}\) and \(s_{n_{k+1}} < \frac{1}{k+1}\).
Just like before, you can’t directly apply the argument above because you don’t know whether \( s_{n_{k+1}} < s_{n_k} \) and you don’t even know whether \( n_{k+1} > n_k \). To get around this, notice that

\[
\min \left\{ \frac{1}{k+1}, s_1, s_2, \ldots, s_{n_k} \right\} > 0 = \inf \{ s_n \mid n \in \mathbb{N} \}
\]

(This is because there are only finitely many terms in the min, and they’re all positive by assumption).

Therefore, by definition of \( \inf \), there is \( s_{n_{k+1}} \) such that

\[
s_{n_{k+1}} < \min \left\{ \frac{1}{k+1}, s_1, s_2, \ldots, s_{n_k} \right\}
\]

Therefore we get \( s_{n_{k+1}} < \frac{1}{k+1}, s_{n_{k+1}} < s_{n_k} \) and finally \( n_{k+1} > n_k \) because \( s_{n_{k+1}} \) is smaller than (hence different from) all the terms preceding it, so it cannot be equal to any of its previous terms (just like the last example). \( \checkmark \)
STEP 3: Therefore, by the inductive construction, we have found a subsequence \((s_{n_k})\) such that \(s_{n_{k+1}} < s_{n_k}\) (decreasing) and \(0 < s_{n_k} < \frac{1}{k}\) for all \(k\) (The \(> 0\) part is by assumption). Therefore, by the squeeze theorem, we get \(\lim_{k \to \infty} s_{n_k} = 0\) □

4. Monotone Subsequence

Video: [Monotone Subsequence](#)

Finally, to prep for the important Bolzano-Weierstraß Theorem for next time, let’s prove the following miraculous fact:

**Theorem:**

Every sequence \((s_n)\) has a monotonic subsequence

(Recall that monotonic means either nondecreasing or nonincreasing)

Note: There absolutely no assumptions about \((s_n)\). It could be divergent, it could be unbounded, it could be wild! That’s what makes this
theorem so powerful!

**Proof:** Neat!

**Definition:**
We say the number $s_n$ is **dominant** if for all $m > n$, $s_n > s_m$.

Kind of like increasing, except here we’re fixing $n$. Think “Everything is going downhill after $s_n$” (kind of like stocks crashing right after you buy it)

**Case 1:** Suppose there are infinitely many dominant terms, let’s denote them (in order) $s_{n_1}, s_{n_2}, \ldots$
Claim: \((s_{n_k}) = (s_{n_1}, s_{n_2}, \ldots)\) is decreasing

But if \(n_{k+1} > n_k\), then since \(s_{n_k}\) is dominant, by definition we have \(s_{n_{k+1}} < s_{n_k}\) ✓

**Case 2:** There are only finitely many dominant terms.

This is lit! We’ll fail so hard at constructing a decreasing sequence that we’re actually constructing an increasing one ⊗
Let $n_1$ be larger than the largest dominant term. Since $n_1$ is not dominant, by definition there must be $n_2 > n_1$ such that $s_{n_2} \geq s_{n_1}$.

Since $n_2$ is not dominant (we have already exceeded the largest dominant term), there must be $n_3 > n_2$ such that $s_{n_3} \geq s_{n_2}$.

Inductively, since $n_k$ is not dominant, there must be $n_{k+1} > n_k$ such that $s_{n_{k+1}} \geq s_{n_k}$.

Therefore we have inductively constructed a subsequence $(s_{n_k})$ such that $s_{n_{k+1}} \geq s_{n_k}$ for all $k$, which means this subsequence is nondecreasing.

\[ \square \]