Claim 1: \( s_n \leq 2 \) for all \( n \)

Proof: Let \( P_n \) be the proposition “\( s_n \leq 2 \)”

Base Case: \( s_1 = 1 \leq 2 \checkmark \)

Inductive Step: Suppose \( P_n \) is true, that is \( s_n \leq 2 \), show \( P_{n+1} \) is true, that is \( s_{n+1} \leq 2 \). But then, we get:

\[
    s_{n+1} = \sqrt{s_n + 1} \leq \sqrt{2 + 1} = \sqrt{3} \leq \sqrt{4} = 2
\]

Hence \( P_{n+1} \) is true, so \( P_n \) is true for all \( n \), that is \( s_n \leq 2 \) for all \( n \) \( \checkmark \)

Claim 2: \((s_n)\) is increasing

Proof: Let \( P_n \) be the proposition \( s_{n+1} > s_n \)

Base Case: \( s_2 = \sqrt{s_1 + 1} = \sqrt{1 + 1} = \sqrt{2} > 1 = s_1 \checkmark \)

Inductive Step: Suppose \( P_n \) is true, that is \( s_{n+1} > s_n \). Show \( P_{n+1} \) is true, that is \( s_{n+2} > s_{n+1} \). But:

\[
    s_{n+2} = \sqrt{s_{n+1} + 1} = \sqrt{\sqrt{s_n + 1} + 1} > \sqrt{2 + 1} = \sqrt{3} > 1 = s_1 \checkmark
\]

Date: Due: Thursday, April 30, 2020.
\[ s_{n+2} - s_{n+1} = \sqrt{s_{n+1} + 1} - \sqrt{s_n + 1} \]
\[ = \left( \sqrt{s_{n+1} + 1} - \sqrt{s_n + 1} \right) \left( \frac{\sqrt{s_{n+1} + 1} + \sqrt{s_n + 1}}{\sqrt{s_{n+1} + 1} + \sqrt{s_n + 1}} \right) \]
\[ = \frac{(\sqrt{s_{n+1} + 1})^2 - (\sqrt{s_n + 1})^2}{\sqrt{s_{n+1} + 1} + \sqrt{s_n + 1}} \]
\[ = \frac{s_{n+1} + 1 - s_n - 1}{\sqrt{s_{n+1} + 1} + \sqrt{s_n + 1}} \]
\[ = \frac{s_{n+1} - s_n}{\sqrt{s_{n+1} + 1} + \sqrt{s_n + 1}} \]

But by the inductive hypothesis, \( s_{n+1} - s_n > 0 \), and the denominator is positive as well, and so \( s_{n+2} - s_{n+1} > 0 \), that is \( s_{n+2} > s_{n+1} \).

Therefore \( P_{n+1} \) is true, and hence \( P_n \) is true for all \( n \), that is \( s_{n+1} > s_n \) for all \( n \) \( \checkmark \).

Therefore since \((s_n)\) is increasing and bounded above (by 2), by the Monotone Sequence Theorem, \((s_n)\) converges to \( s \).

Passing to the limit in the identity \( s_{n+1} = \sqrt{s_n + 1} \) we get:

\[ s = \sqrt{s + 1} \]
\[ s^2 = s + 1 \]
\[ s^2 - s - 1 = 0 \]
\[ s = \frac{1 \pm \sqrt{5}}{2} \]
But $\frac{1-\sqrt{5}}{2} < 0$, but $s_n \geq 0$ for all $n$ and therefore $s = \frac{1+\sqrt{5}}{2} = \phi$ (the golden ratio).

**AP 2**

First of all, $t_n > 0$ for all $n$ (easy induction). Moreover, for all $n$, 

$$\frac{t_{n+1}}{t_n} = \frac{(\frac{n}{n+2}) t_n}{t_n} = \frac{n}{n+2} < 1$$

And therefore $t_{n+1} < t_n$, so $(t_n)$ is decreasing.

Since $(t_n)$ is decreasing and bounded below, by the Monotone Sequence Theorem, $(t_n)$ converges.

**AP 2**

Consider $S = \{s_n \mid n \in \mathbb{N}\}$.

Since $(s_n)$ is bounded below, for some $C$, we have $s_n \geq C$ for all $n$, and therefore $S$ is bounded below and therefore has a least upper bound $s = \inf(S)$

**Claim:** $(s_n)$ converges to $s$.

Let $\epsilon > 0$ be given, then notice $s + \epsilon > m$ and therefore there is some $s_N$ such that $s_N < s + \epsilon$. But then, since $(s_n)$ is decreasing, if $n > N$, $s_n < s_N < s + \epsilon$, so $s_n < s + \epsilon$, so $s_n - s < \epsilon$. Moreover, since $(s_n)$ is bounded below by $s$, we also have $s_n \geq s > s - \epsilon$, so $s_n - s > -\epsilon$.
Therefore, if \( n > N \), \(-\epsilon < s_n - s < \epsilon\), so \(|s_n - s| < \epsilon\).

Therefore \((s_n)\) converges to \(m\) \(\square\)

**AP 4**

Let \((s_n)\) be a Cauchy sequence in \(\mathbb{Z}\). We want to show that \((s_n)\) converges.

But letting \(\epsilon = 1\) in the definition of Cauchy, there is \(N\) such that if \(m, n > N\), then \(|s_n - s_m| < 1\).

But since there are no different integers that are at most 1 apart, we get that if \(m, n > N\), then \(s_n = s_m\), and so \((s_n)\) is just the sequence \((s_1, s_2, \ldots, s_N, s_{N+1}, s_{N+1}, s_{N+1}, \ldots)\), which is eventually constant and therefore converges to \(s_{N+1}\).

**AP 5**

**Reflexivity:** Is \((p_n) \sim (p_n)\)? Yes because \(p_n - p_n = 0\), and therefore

\[
\lim_{n \to \infty} p_n - p_n = \lim_{n \to \infty} 0 = 0
\]

**Symmetry** Suppose \((p_n) \sim (q_n)\), that is \(\lim_{n \to \infty} |p_n - q_n| = 0\), but then

\[
\lim_{n \to \infty} q_n - p_n = -\lim_{n \to \infty} |p_n - q_n| = -0 = 0
\]

Hence \((q_n) \sim (p_n)\) \(\checkmark\)

**Transitivity** Suppose \((p_n) \sim (q_n)\) and \((q_n) \sim (r_n)\). But then
\[
\lim_{n \to \infty} p_n - r_n = \lim_{n \to \infty} p_n - q_n + q_n - r_n \\
= \lim_{n \to \infty} p_n - q_n + \lim_{n \to \infty} q_n - r_n \\
= 0 + 0 \\
= 0
\]

Hence \((p_n) \sim (r_n) \checkmark\)

**AP 6**

Suppose \((s_n)\) doesn’t converge to \(s\). Then there is \(\epsilon > 0\) such that for all \(N\) there is \(n > N\) with \(|s_n - s| > \epsilon\)

In particular, for \(N = 1\) there is \(n_1 > 1\) with \(|s_{n_1} - s| > \epsilon\)

And for \(N = n_1\) there is \(n_2 > n_1\) with \(|s_{n_2} - s| > \epsilon\)

Continuing in this fashion, we get a subsequence \((s_{n_k})\) of \((s_n)\) with \(|s_{n_k} - s| > \epsilon\) for all \(k\). But no subsequence of \((s_{n_k})\) converges to \(s\) (since all the terms are at least \(\epsilon\) away from \(s\)) \(\square\)

**AP 7(a)**

Suppose \(s_n\) doesn’t converge to \(s\). Then there is \(\epsilon > 0\) such that for all \(N\) there is \(n > N\) with \(|s_n - s| > \epsilon\). Construct the same subsequence \((s_{n_k})\) as in the previous problem.

Since \((s_{n_k})\) is bounded, by the Bolzano-Weierstraß Theorem, \((s_{n_k})\) has a convergent subsequence. By assumption, that (sub-)subsequence must converge to \(s\), but this contradicts \(|s_{n_k} - s| > \epsilon\) for all \(k\) (and in particular for the sub-subsequence) \(\Rightarrow \Leftarrow\) \(\square\)
AP 7(B)

Consider:

\[ s_n = \begin{cases} 
  n & \text{if } n \text{ is even} \\
  0 & \text{if } n \text{ is odd} 
\end{cases} \]

Then every convergent subsequence of \((s_n)\) must converge to 0 (because it must eventually end with all 0’s), but \((s_n)\) doesn’t converge to 0.

AP 8(A)

**Claim 1:** \( s_n \geq \sqrt{a} \) for all \( n \)

First of all, \( s_1 = b \geq \sqrt{a} \). Moreover, if \( m > 1 \), then \( m = n + 1 \) for some \( n \in \mathbb{N} \), and so:

**Inductive Step:** Suppose \( P_n \) is true, that is \( s_n \geq \sqrt{a} \). Show \( P_{n+1} \) is true, that is \( s_{n+1} \geq \sqrt{a} \), but then:

\[
\begin{align*}
    s_m - \sqrt{a} &= s_{n+1} - \sqrt{a} \\
    &= \frac{1}{2} \left( s_n + \frac{a}{s_n} \right) - \sqrt{a} \\
    &= \frac{1}{2} \left( s_n - 2\sqrt{a} + \frac{a}{s_n} \right) \\
    &= \frac{1}{2} \left( \sqrt{s_n} \right)^2 - 2\sqrt{s_n} \left( \frac{\sqrt{a}}{s_n} \right) + \left( \frac{\sqrt{a}}{s_n} \right)^2 \\
    &= \frac{1}{2} \left( \sqrt{s_n} - \frac{\sqrt{a}}{s_n} \right)^2 \\
    &\geq 0
\end{align*}
\]
This combined with $s_1 \geq \sqrt{a}$ allows us to conclude that for all $n \in \mathbb{N}$, we have $s_n \geq \sqrt{a}$.

**Claim 2:** $s_{n+1} \leq s_n$ for all $n$

But

$$s_{n+1} - s_n = \frac{1}{2} \left( s_n + \frac{a}{s_n} \right) - s_n$$
$$= \frac{1}{2} \left( -s_n + \frac{a}{s_n} \right)$$
$$= \frac{1}{2} \left( -\left( s_n \right)^2 + a \right)$$

But from before $s_n \geq \sqrt{a}$, so $(s_n)^2 \geq a$ and so $-(s_n)^2 + a \leq 0$

And therefore we get $s_{n+1} - s_n \leq 0$, so $s_{n+1} \leq s_n$

Therefore $(s_n)$ is a nonincreasing sequence that is bounded below by $\sqrt{a}$, and hence $(s_n)$ converges to $s$.

And passing to the limit in

$$s_{n+1} = \frac{1}{2} \left( s_n + \frac{a}{s_n} \right)$$

We get:
\[
s = \frac{1}{2} \left( s + \frac{a}{s} \right)
\]
\[
2s = \frac{s^2 + a}{s}
\]
\[
2s^2 s^2 + a
\]
\[
s^2 = a
\]
\[
s = \pm \sqrt{a}
\]

But since \( s_n \geq 0 \), we ultimately get \( s = \sqrt{a} \).

**AP 8(b)**

\[
s_1 = \frac{1}{2}
\]
\[
s_2 = \frac{1}{2} \left( 2 + \frac{2}{2} \right) = \frac{3}{2} = 1.5
\]
\[
s_3 = \frac{1}{2} \left( 1.5 + \frac{2}{1.5} \right) \approx 1.41666 \ldots
\]
\[
s_4 = \frac{1}{2} \left( 1.41666 + \frac{2}{1.41666} \right) \approx 1.4142 \ldots
\]

**AP 9**

(1) In class, we proved the Monotone Sequence Theorem, using the Least Upper Bound Property, which we assumed is true ever since section 4. In particular, the proof relies on using \( \text{sup}(S) \), which uses the LUB Property.

(2) If there is no such \( m \), then for all \( a \in S \) and all \( m \in \mathbb{N} \), \( a \leq s_n + 2^{-m} \), so \( a \leq s_n \), but then \( s_n \) is the largest element of \( S \), and so \( s_n = \max(S) \), and therefore \( S \) has a least upper bound, namely \( s_n \) (and we’d be done)
(3) By assumption $s_{n+1} > s_n + 2^{-m}$, so $s_{n+1} - s_n > 2^{-m} > 0$, so $s_n$ is increasing.

(4) $(s_n)$ is increasing and bounded above (since $S$ is bounded above), so $(s_n)$ converges by the Monotone Sequence Theorem.

(5) This follows because $\lim_{k \to 2^{-n}} = 0$ (which doesn’t use the Least Upper Bound Property). In particular, with $\epsilon = a - s > 0$, we know there is $N$ such that if $k > N$, then $2^{-k} < a - s$.

(6) Use the definition of $(s_n)$ converges to $s$ but with $\epsilon = 2^{-k-1}$, so there is $N$ such that if $m > N$ then $|s_m - s| < 2^{-k-1}$, so $s_m - s > -2^{-k-1}$ so $s - 2^{-k-1} < s_m$. And $s_m < s$ follows because $(s_n)$ is increasing with limit $s$ (and $s_m \neq s$ otherwise we would be done).

(7) Since $s_m < s$, $-s_m > -s$, so $a - s_m > a - s > 2^{-k}$, so $a > s_m + 2^{-k}$ and therefore by definition of $s_{m+1}$, $s_{m+1} > s_m + 2^{-k}$.

(8) This is a contradiction with the fact that $(s_n)$ is increasing with limit $s$.

(9) Then $s_n - s > -(s - s')$ so $s_n - s > s' - s$ so $s_n > s'$, so we found an element in $S$ (namely $s_n$) that is bigger than $s'$, which contradicts the fact that $s'$ is an upper bound of $S$ $\implies \square$. 