HOMEWORK 3 – SELECTED BOOK SOLUTIONS

6.2

First of all, by definition of a cut, we know there is $r_1 \in \alpha$ and there is $r_2 \in \beta$, and so by definition $r_1 + r_2 \in \alpha + \beta$, so $\alpha + \beta$ is nonempty ✓

Since $\alpha \neq \mathbb{Q}$ there is $m_1 \notin \alpha$ in particular, $r_1 < m_1$ for every $r_1 \in \alpha$, otherwise $m_1 \leq r_1$ and by definition of a cut we would have $m_1 \in \alpha$. Similarly there is $m_2 \notin \beta$ such that $r_2 < m_2$ for all $r_2 \in \beta$. But then for all $r = r_1 + r_2 \in \alpha + \beta$, we have $r = r_1 + r_2 < m_1 + m_2$ and in particular $m_1 + m_2 \notin \alpha + \beta$. Therefore $\alpha + \beta \neq \mathbb{Q}$ ✓

Suppose $r \in \alpha + \beta$ and $s < r$ is rational, then by definition $r = r_1 + r_2$ where $r_1 \in \alpha$ and $r_2 \in \beta$, then $s < r \Rightarrow s < r_1 + r_2 \Rightarrow s - r_2 < r_1$ so since $r_1 \in \alpha$ and $\alpha$ is a cut, we have $s - r_2 \in \alpha$ and so

$$s = (s - r_2) + r_2 \in \alpha + \beta$$

Suppose $r \in \alpha + \beta$, then $r = r_1 + r_2$ where $r_1 \in \alpha$ and $r_2 \in \beta$, then since $\alpha$ is a cut there is $s_1 \in \alpha$ with $s_1 > r_1$ and since $\beta$ is a cut there is $s_2 \in \beta$ with $s_2 > r_2$, but then if $s =: s_1 + s_2$ then $s \in \alpha + \beta$ and

$$r = r_1 + r_2 < s_1 + s_2 = s$$

✓ □

7.4a Let $x_n = \frac{\sqrt{2}}{n}$ which are irrational, because otherwise $nx_n = \sqrt{2}$ would be rational (being the product of two rational numbers), but

\[ Date: Thursday, April 16, 2020. \]
\[ \lim_{n \to \infty} x_n = 0, \text{ which is rational} \]

**7.4b** Let \( x_n = \left(1 + \frac{1}{n}\right)^n \), which is rational (being the product of rational numbers), but \( \lim_{n \to \infty} x_n = e \), which is irrational.

**Note:** Another solution would be \( x_n = \) expansion of \( \pi \) up to the \( n \)th decimal place, like \( x_1 = 3.1, x_2 = 3.14, x_3 = 3.141 \) then each \( x_n \) is rational but \( x_n \) converges to \( \pi \), which is irrational.

**8.4**

**Scratchwork:**

\[ |s_n t_n| = |s_n| |t_n| \leq |s_n| M < \epsilon \Rightarrow |s_n| < \frac{\epsilon}{M} \]

**Proof:** Let \( \epsilon > 0 \) be given. Since \( s_n \to 0 \) we know that there is an \( N \) such that for all \( n \), if \( n > N \), then \( |s_n| < \frac{\epsilon}{M} \). With that same \( N \), if \( n > N \), we therefore get:

\[ |s_n t_n| = |s_n| |t_n| \leq |s_n| M < \left(\frac{\epsilon}{M}\right) M = \epsilon \checkmark \]

**8.5b** In my opinion, this is easier to prove directly:

Let \( \epsilon > 0 \) be given, then there is \( N \) such that if \( n > N \), then \( |t_n| = t_n < \epsilon \). With that same \( N \), if \( n > N \), then

\[ |s_n| \leq t_n < \epsilon \checkmark \]

**8.7b** Suppose \( \lim_{n \to \infty} \sin \left(\frac{\pi n}{3}\right) = a \). Then for all \( \epsilon > 0 \) there is \( N \) such that if \( n > N \), then

\[ \left| \sin \left(\frac{\pi n}{3}\right) - a \right| < \epsilon \]

But now let \( n \) be any integer \( > N \) that is a multiple of 6, then
\[ \left| \sin \left( \frac{\pi n}{3} \right) - a \right| = |a| < \epsilon \]

So \(-\epsilon < a < \epsilon\)

Similarly, let \(n\) be any integer \(> N\) that is congruent to 1 modulo 6, then

\[ \left| \sin \left( \frac{\pi n}{3} \right) - a \right| = \left| \sqrt{3} - a \right| = \left| a - \frac{1}{2} \right| < \epsilon \]

So \(-\epsilon < a - \sqrt{3} < \epsilon \Rightarrow \sqrt{3} - \epsilon < a < \sqrt{3} + \epsilon\).

Now choose \(\epsilon > 0\) such that \(\epsilon \leq \frac{\sqrt{3}}{2} - \epsilon\), that is \(\epsilon \leq \frac{\sqrt{3}}{4}\), then we get the contradiction:

\[ a < \epsilon \leq \frac{\sqrt{3}}{2} - \epsilon < a \Rightarrow a < a \Rightarrow \square \]

Hence \(\lim_{n \to \infty} \sin \left( \frac{\pi n}{3} \right)\) doesn’t exist.

8.8a

**Scratchwork:** Notice that
\[
\left| \sqrt{n^2 + 1} - n - 0 \right| = \left| \sqrt{n^2 + 1} - n \right|
\]
\[
= \left| (\sqrt{n^2 + 1} - n) \left( \frac{\sqrt{n^2 + 1} + n}{\sqrt{n^2 + 1} + n} \right) \right|
\]
\[
= \frac{n^2 + 1 - n^2}{\sqrt{n^2 + 1} + n}
\]
\[
= \frac{1}{\sqrt{n^2 + 1} + n}
\]
\[
\geq 0
\]
\[
< \frac{1}{n} < \epsilon
\]

Which implies \( n > \frac{1}{\epsilon} \).

**Proof:** Let \( \epsilon > 0 \) be given, and let \( N = \frac{1}{\epsilon} \). Then if \( n > N \), then we get

\[
\left| \sqrt{n^2 + 1} - n - 0 \right| = \left| \sqrt{n^2 + 1} - n \right|
\]
\[
= \frac{1}{\sqrt{n^2 + 1} + n}
\]
\[
= \frac{1}{\sqrt{n^2 + 1} + n}
\]
\[
< \frac{1}{n}
\]
\[
= \epsilon \square
\]
8.9.a Let $\epsilon > 0$ be arbitrary, then there is $N$ such that if $n > N$, then $|s_n - s| < \epsilon$, where $s = \lim_{n} s_n$.

However:

$$|s_n - s| < \epsilon \Rightarrow |s - s_n| < \epsilon \Rightarrow -\epsilon < s - s_n < \epsilon \Rightarrow s > s_n - \epsilon$$

In particular, if you pick $n$ such that $n > N$ and $s_n \geq a$ (which we can since there are only finitely many $n$ such that $s_n < a$), $s > s_n - \epsilon \geq a - \epsilon$, hence $s > a - \epsilon$.

Since $\epsilon$ was arbitrary, we ultimately get $s \geq a$. \qed

Note: If you want to make that last step more elegant, suppose $s < a$ and choose $\epsilon$ such that $a - \epsilon > s$, that is $\epsilon < a - s$, then $s > a - \epsilon > s \Rightarrow s > s \Rightarrow \Leftarrow$.

8.10 Let $\epsilon > 0$ be TBA, then since $\lim_{n \to \infty} s_n = s$, there is $N$ such that if $n > N$, then $|s_n - s| < \epsilon$.

But

$$|s_n - s| < \epsilon \Rightarrow -\epsilon < s_n - s < \epsilon \Rightarrow s - \epsilon < s_n < s + \epsilon$$

In particular, we get $s_n > s - \epsilon$

Now choose $\epsilon$ such that $s - \epsilon \geq a$, that is $\epsilon \leq s - a$ (Which we can do since $s - a > 0$ by assumption), then for $n > N$, we get

$$s_n > s - \epsilon \geq a$$

So $s_n > a$ for $n > N$. \qed