**MATH 140A – HOMEWORK 2**

**Readings:** Sections 3 and 4. In section 3, you don’t need to memorize the field and ordered field axioms, but know how to prove all the properties in that section. The Triangle Inequality is very important. Section 4 is very hard but the most important section in the book, make sure to understand every single detail in it.

- **Section 3:** 5, 6b, 8 (Hint: By Contradiction), AP1, AP2, (Optional: AP5)
- **Section 4:** 7, 8, 14, 16, AP3, AP4 (Optional: AP6)

**Additional Problem 1:** Here are some applications of the triangle inequality in real analysis life. I highly recommend reviewing the proof of the corollary given on page 12 of my Lecture 3 notes.

(a) Show that if $|x - y| < \frac{\epsilon}{2}$ and $|y| < \frac{\epsilon}{2}$, then $|x| < \epsilon$

(b) Show that if $|x - y| < \frac{\epsilon}{3}$ and $|y - z| < \frac{\epsilon}{3}$ and $|z - t| < \frac{\epsilon}{3}$, then $|x - t| < \epsilon$

(c) Show if $|x - z| \leq \frac{\epsilon}{2}$ and $|y - z| \geq \epsilon$, then $|y - x| \geq \frac{1}{2} |y - z|$.

**Hint for (c):** Start with $|y - z|$ and write $y - z$ in terms of $y - x$ and $x - z$, and then use $\epsilon \leq |y - z|$.

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Date: Due: Thursday, April 9, 2020.

1In the second edition, there is an additional hint: To show $\sup(A) + \sup(B) \leq \sup(S)$, show that for each $b \in B$, $\sup(S) - b$ is an upper bound for $A$, hence $\sup(A) \leq \sup(S) - b$. Then show $\sup(S) - \sup(A)$ is an upper bound for $B$.
Additional Problem 2: For this, problem review the Reverse Triangle Inequality at the end of the Lecture 3 notes. Explain your answers.

(a) Suppose $|x - a| < 4$, then what can you say about $||x| - |a||$?

(b) How small should $|x - a|$ be in order to guarantee that $||x| - |a|| < 2$?

(c) Find $\delta > 0$ such that if $|x - a| < \delta$, then $||x| - |a|| < \frac{1}{2}$

(d) Finally, generalize the result in (c) as follows: if $\epsilon > 0$ is fixed, can you find $\delta > 0$ such that if $|x - a| < \delta$, then $||x| - |a|| < \epsilon$?

Congratulations, you have just shown that $f(x) = |x|$ is continuous!

Additional Problem 3: Use the definition of sup and inf to find the following. Explain why your answer is correct.

(a) sup($A$) where $A = \{3 - \frac{2}{n}, n \in \mathbb{N}\}$

(b) inf($B$) where $B = \{e^{-x}, x \in \mathbb{R}\}$

(c) sup($C$) where $C = \{n(-1)^n, n \in \mathbb{N}\}$

Hint: Is $C$ bounded above? Why or why not?

Additional Problem 4: Let $A$ and $B$ be nonempty bounded subsets of $\mathbb{R}$ and let $AB$ be the set of all products $ab$ where $a \in A$ and $b \in B$. Is it always true that sup($AB$) = sup($A$) sup($B$)?

Optional Additional Problem 5: Show that there is no order structure $\leq$ on $\mathbb{C}$ that agrees with the standard order structure $\leq$ on $\mathbb{R}$.
**Hint:** Consider two cases: $i \geq 0$ and $i \leq 0$.

**Note:** In fact, more is true! There is no order structure on $\mathbb{C}$ at all! See this video if you’re interested: [Can you compare complex numbers?](#)

**Optional Additional Problem 6:**

**Note:** I highly encourage you to do this problem; in fact I was this close to making this non-optional, but I decided to go easy on you! This problem gives you a taste of what an actual real analysis proof looks like.

Fill in the gaps in the following proof that $\sqrt{x}$ exists. The gaps that you have to fill out are written in blue

**Goal:** Show that for every $x > 0$ and every $n > 0$ there is a unique $y > 0$ such that $x^n = y$

**STEP 1:** Fix $x > 0$ and define $S$ as:

$$S = \{ t > 0 \mid t^n < x \}$$

For this step *only*, let $t = \frac{x}{x+1}$

**1.** Show using the definition of $t$ that $0 < t < 1$

Since $0 < t < 1$, we have $t^n < t$

**2.** Show using the definition of $t$ that $t < x$

**3.** Conclude that $S \neq \emptyset$
STEP 2: Now if $t > 1 + x$, then, since $t > 1$ we have $t^n > t$

(4) Show $t > x$, and then conclude $t \notin S$

(5) Conclude that $1 + x$ must be an upper bound for $S$

(6) Explain why $S$ must have a least upper-bound (This is quick)

Let’s call that least upper bound $y$

To prove $y^n = x$ we will show that we cannot have $y^n < x$ and we cannot have $y^n > x$

STEP 3:

Consider for all $a, b$ the identity (do not prove this)

$$b^n - a^n = (b - a) (b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$$

(7) Use the above to show that if $0 < a < b$, then $b^n - a^n < (b - a) nb^{n-1}$

Now assume $y^n < x$. Choose $h$ such that both $0 < h < 1$ and

$$h < \frac{x - y^n}{n(y + 1)^{n-1}}$$

(8) Why can we choose such a $h$? (also quick)

(9) By letting $a = y$ and $b = y + h$ in $b^n - a^n < (b - a) nb^{n-1}$, show

$$(y + h)^n - y^n < hn(y + h)^{n-1} < hn(y + 1)^{n-1} < x - y^n$$
(10) Conclude \( y + h \in S \). Why is this a contradiction? \( \Rightarrow \Leftarrow \)

Therefore \( y^n \geq x \)

**STEP 4:** Now assume \( y^n > x \) and consider

\[
k = \frac{y^n - x}{ny^{n-1}}
\]

(11) Show \( 0 < k < y \)

Suppose \( t \geq y - k \)

(12) Show, using the identity \( b^n - a^n < (b - a)nb^{n-1} \) that

\[
y^n - t^n \leq y^n - (y - k)^n < kny^{n-1} = y^n - x
\]

(13) Conclude \( t^n > x \) and therefore \( t \notin S \).

(14) Deduce that \( y - k \) is an upper bound for \( S \) and find a contradiction \( \Rightarrow \Leftarrow \)

Hence we get \( y^n = x \)

**STEP 5:** Uniqueness:

Suppose \( y_1 > 0 \) and \( y_2 > 0 \) solve \( (y_1)^n = x \) and \( (y_2)^n = x \) but \( y_1 \neq y_2 \)

(15) Use the identity below to find a contradiction

\[
b^n - a^n = (b - a) \left( b^{n-1} + b^{n-2} + \cdots + a^{n-1} \right)
\]

And you’re done! \( \Box \)