MATH 2B – COMPARISON THEOREM

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If \(0 \leq f \leq g\) and \(\int_a^\infty g(x)\,dx < \infty\), then \(\int_a^\infty f(x)\,dx < \infty\) as well.

**Example:** Show that \(\int_0^\infty e^{-x^2}\,dx\) is convergent.

**Step 1:** First, split up the integral as:

\[
\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx
\]

**Step 2:** For the first integral, on \([0, 1]\), we have \(-x^2 \leq 0\), so \(e^{-x^2} \leq 1\), but then

\[
\int_0^1 e^{-x^2}\,dx \leq \int_0^1 1\,dx = 1 < \infty
\]

So the first integral is convergent.

**Step 3:** For the second integral, on \([1, \infty)\), we have \(x \geq 1\), so \(x^2 \geq x\) (multiply both sides by \(x > 0\)), so \(-x^2 \leq -x\), so \(e^{-x^2} \leq e^{-x}\) and so

\[
\int_1^\infty e^{-x^2}\,dx \leq \int_1^\infty e^{-x}\,dx = \left[-e^{-x}\right]_1^\infty = \lim_{x \to \infty} -e^{-x} = -0 + e^{-1} = e^{-1}
\]

Therefore the second integral is convergent.

**Step 4:** Finally \(\int_0^\infty e^{-x^2}\,dx\) is convergent because both pieces are convergent.

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\^There are no discontinuities here, but it’ll make our problem easier