The theory of elasticity is to study the deformation of elastic solid bodies under external load. A body is elastic if, when the external forces are removed, the bodies return to their original (undeformed) shape.

1. Introduction

How to describe the deformation of a solid body? Let us denoted a solid body by a bounded domain $\Omega \subset \mathbb{R}^3$. Then the deformed domain can be described as the image of a vector-valued function $\Phi : \Omega \rightarrow \mathbb{R}^3$, i.e. $\Phi(\Omega)$, which is called a configuration or a placement of a body. For most problems of interest, we can require that the map be 1-1 and differentiable. Since we are interested in the deformation, i.e., the change of the domain, we let $\Phi(x) = x + u(x)$ or equivalently $u(x) = \Phi(x) - x$ and call $u$ the displacement. Here we deal with continuum mechanics, i.e., systems have properties defined at all points in space, ignoring details in atom and molecules level.

The displacement is not the deformation. For example, a translation or a rotation, so-called rigid body motions, of $\Omega$ will lead to a non-trivial displacement $u \neq 0$, but the shape and volume of $\Omega$ does not change at all. How to describe the shape mathematically? Vectors. For example, a cube can be described by 3 orthogonal vectors. The change of the
shape will then be described by a mapping of these three vectors, which can be represented by a $3 \times 3$ matrix. For example, a rigid motion has the form
\begin{equation}
\Phi(x) = x_1 + Q(x - x_0),
\end{equation}
where $x_0, x_1$ are fixed material points and $Q$ is a rotation (unitary) matrix ($Q^T Q = I$).

Consider a point $x \in \Omega$ and a vector $z$ pointing from $x$ to $x + z$. Then the vector $z$ will be deformed to $\Phi(x + z) - \Phi(x) \approx \nabla \Phi(x) z$ provided $\|z\|$ is small. So the deformation gradient $F(x) := \nabla \Phi(x)$ is a candidate of the mathematical quantity to describe the deformation.

The deformation gradient can be decomposed into the summation of a symmetric part and an anti-symmetric part. The symmetric part will ... the change of the geometric metric and the antisymmetric part is for the rotation.

Let us compute the change of the distance. For two points $x$ and $x + z$ in $\Omega$, the squared distance in the deformed domain will be
\begin{equation}
\|\Phi(x + z) - \Phi(x)\|^2 \approx \|\nabla \Phi(x) z\|^2 = z^T F^T F z.
\end{equation}

Comparing with the squared distance $\|x + z - x\|^2 = z^T I z$ in the original domain, we then define a symmetric matrix function
\begin{equation}
E = \frac{1}{2} (F^T F - I)
\end{equation}
and call it strain. Very crudely, strain may be thought of as a relative displacement per unit distance between two points in a body. When $\Phi$ is a rigid body motion, $F = Q$ with $Q^T Q = I$, and thus $E = 0$. A nonzero strain $E$ will describe change in shape.

Strain is the geometrical measure of the deformation representing the relative displacement between points in the material body, i.e. a measure of how much a given displacement differs locally from a rigid-body displacement. This is the right quantity to describe the deformation.

What is the relation between the strain $E$ and the displacement $u$? Substituting $\nabla \Phi = I + \nabla u$ into (2), we get
\begin{equation}
E = E(u) = \frac{1}{2} (\nabla u + (\nabla u)^T + (\nabla u)^T \nabla u).
\end{equation}

We shall mostly studied the small deformation case, i.e. $\nabla u$ is small. Then we can drop the quadratic part and approximately use
\begin{equation}
E(u) \approx \varepsilon(u) = \nabla^s u := \frac{1}{2} (\nabla u + (\nabla u)^T),
\end{equation}
where $\nabla^s$ will be called the symmetric gradient. The strain tensor $E$ defined in (3) is called the Green-St. Venant Strain Tensor or the Lagrangian Strain Tensor. Its approximation $\varepsilon$ in (4) is called the infinitesimal strain tensor.

Remark 1.1. The approximation $E(u) \approx \varepsilon(u)$ is valid only for small deformation. Consider the rigid motion (1). The deformation gradient is $F = Q$ and thus $E(u) = 0$ but $\varepsilon(u) = (Q + Q^T)/2 - I \neq 0$. Hence the infinitesimal strain tensor $\varepsilon$ does not measure the correct strain when there are large rotations though the strain tensor $E$ can.

When considering the linear elasticity theory, we also call the kernel of $\varepsilon$ rigid body motion. Consider the rotation in 2-D which can be written as $\Phi(x) = e^{i\theta} x$. Then $u(x) = (e^{i\theta} - 1)x = i \theta x + O(\theta^2)$. Skipping the high order term when $|\theta| \ll 1$, we obtain the rigid motion for $\varepsilon$ is a rotation of $x = (x, y)$ by $90^\circ$ counterclockwise, i.e. $x^\perp := (-y, x)$.

Exercise 1.2. Derive the rigid body motion for $\varepsilon$, i.e., the ker of $\varepsilon$ in 3-D.
How to find out the deformation? To cause a deformation, there should be forces applied to the solid body. In the static equilibrium, the deformation should be determined by the balance of forces. We now discuss possible external and internal forces.

There are two types of external forces. **Body forces** are applied forces distributed throughout the body, e.g. the gravity or magnetic force. **Surface forces** are pressures applied to the surface or outer boundary of the body, such as pressures from applied loads.

There are also internal forces. The deformation of a body will induces resistance forces like the forces produced when we stretch the spring or compress the rubber. How to describe such internal forces? Let us conceptually split the body along a plane through the body. Then we have a new internal surface across which further internal forces are exerted to keep the two parts of the body in equilibrium. These internal forces are called *stresses*.

Mathematically we can define the stress at a point \( x \) as \( \lim_{A \to \{x\}} p_A / A \), where \( p_A \) is the pressure applied to the cutting surface \( A \), which contains \( x \) and shrinks to \( x \). The pressure \( p_A \) is a force (per unit area) which is not necessarily orthogonal to the surface. The normal part is called the *normal stress* and the tangential part is *shearing stress*.

At a fixed point \( x \) of a body there are infinity many of different surface elements with centroid at \( x \). Therefore the stress can be described as a function \( t: \Omega \times S^2 \to \mathbb{R}^3 \), where \( S^2 \) is the unit 2-sphere used to representing a direction. At given a point \( x \in \Omega \), the stress \( t \) maps a unit vector \( n \in S^2 \) to another vector (the force). A theorem by Cauchy simplifies the formulation of the stress function, by separating the variables, as

\[
t(x, n) = \Sigma(x) n,
\]
where the matrix function \( \Sigma(x) \) is called the *stress tensor*.

Suppose \( f \) is the only applied external and body-type force. We can then write out the equilibrium equations

\[
\begin{align*}
\int_V f \, dx + \int_{\partial V} \Sigma \ n \, dS &= 0 \quad \text{for all } V \subset \Omega, \\
\int_V x \times f \, dx + \int_{\partial V} x \times (\Sigma \ n) \, dS &= 0 \quad \text{for all } V \subset \Omega.
\end{align*}
\]

Equation (5) is the balance of forces and (6) is the conservation of angular momentum. It can be derived from (6) that the stress tensor \( \Sigma \) is symmetric. Otherwise the body would be subjected to unconstrained rotation.

**Exercise 1.3.** *Show that equation (6) implies that \( \Sigma \) is symmetric.*

Using the Gauss theorem and letting \( V \to \{x\} \), the equation (5) for the balance of forces becomes

\[
f + \text{div} \Sigma = 0.
\]

The \( 3 \times 3 \) matrix function \( \Sigma \) has 9 components while (5)-(6) contains only 6 equations. Or the symmetric function \( \Sigma \) has 6 components while (7) contains only 3 equations. We need more equations to determine \( \Sigma \) uniquely.

The internal force is a consequence of the deformation. Therefore it is reasonable to assume the stress is a function of the strain and in turn a function of the displacement, i.e.

\[
\Sigma = \Sigma(E) = \Sigma(u),
\]
which is called *constitutive equation*. Then by substituting (8) into (7), it is possible to solve 3 equations to get 3 unknowns (3 components of the vector function \( u \)).

It is then crucial to figure out the relation \( \Sigma(E) \) in (8). First we assume the stress tensor is intrinsic, i.e., independent of the choice of the coordinate. Second we consider
the isotropic material, i.e., the stress vectors do not change if we rotate the non-deformed body. With such hypothesis, we can assume
\[ \Sigma = \Sigma(I + 2E). \]

By Rivlin-Ericksen theorem, we can then expand it as
\[ \Sigma = \lambda \text{trace}(E)I + 2\mu E + o(E), \]
where \( \lambda, \mu \) are two positive constants, known as Lamé constants. For the theory of linear elasticity, we drop the high order term and use the approximation
\[ \Sigma \approx \sigma = \lambda \text{trace}(E)I + 2\mu E, \]
which is appropriate for
- small strain known as Hook’s law for linear material.
- St. Venant-Kirchhoff material (not just for small strain).

We now summarize the unknowns and equations for linear elasticity as the follows.

- \( (\sigma, \varepsilon, u) \): (stress, strain, displacement).
- \( \sigma, \varepsilon \) are symmetric \( 3 \times 3 \) matrix functions and \( u \) is a \( 3 \times 1 \) vector function.
- Kinematic equation
  \[ \varepsilon = \nabla \times u = \frac{1}{2}(\nabla u + (\nabla u)^T). \]
- Constitutive equation
  \[ \sigma = \lambda \text{trace}(\varepsilon)I + 2\mu \varepsilon = \lambda \text{div} u + 2\mu \nabla \times u. \]
- Balance equation
  \[ f + \text{div} \sigma = 0. \]
\( e_1^T E e_2 \approx \varepsilon_{12} \). On the other hand, let us calculate the change of angles. The original angle between \( e_1 \) and \( e_2 \) is \( \pi/2 \). The angle of the deformed vectors is denoted by \( \theta \) and the change of the angle by \( \delta \theta \), i.e., \( \theta = \pi/2 - \delta \theta \). Then

\[
2(Ee_1, e_2) = (Fe_1, Fe_2) = \|Fe_1\|\|Fe_2\| \cos \theta = (1 + \delta e_1)(1 + \delta e_2) \sin \delta \theta \sim \delta \theta.
\]

In the \( \sim \), we use the linear approximation \( \sin \delta \theta \sim \theta \) and skip the quadratic and higher order terms since the change in length and angle are small. In summary, \( \varepsilon_{12} = \delta \theta / 2 \), i.e., the shear strain \( \varepsilon_{ij} \) represents the change of angles and \( \varepsilon_{ii} \) represents the change of length in the \( i \)-th coordinate direction.

Now we understand the stress. Consider the deformation of a bar-like body. On the conceptually cutting plane, the deformation will introduce two types for stress

- **normal stress** which is normal to the plane;
- **shear stress** which is tangential to the plane.

The normal stress is considered as positive if it produces tension, and negative if it produces compression.

Normal or shear is a relative concept. For a horizontally stretched bar, there is no shear stress if the cutting plane is vertical or no normal stress if the cutting plane is horizontal. For a symmetric and positive definite matrix, we can use eigen-vectors to form an orthonomal basis and in the new coordinate, the stress will be diagonal. The eigen-vectors of the stress tensor will be called principal directions, which are functions of locations. Then along the principal directions, we only see the normal stress. Similarly there are principle directions for strain tensor. Note that strain and stress tensor do not necessarily share the same eigenvectors.

Both stress and strain can be represented by symmetric matrix functions. We emphasize that as intrinsic quantities of the material, they should not depend on the coordinate. In other words, these matrix representation should follow certain rules when we change the coordinate. We shall explore more in the Section 3 on Tensor.

How to interpret the constitutive equation relating stress and strain? We first derive a simple relation between the normal stress and the normal strain. Consider a spring elongated in the \( x \)-direction. By Hook’s law, the stress induced by the elongation will be proportional to the strain, i.e.,

\[
\sigma_{11} = E \varepsilon_{11} \text{ or equivalently } \varepsilon_{11} = \frac{1}{E} \sigma_{11}.
\]

The positive parameter \( E \) is called the modulus of elasticity in tension which is a property of the material. Usually \( E \approx 10^{11} \) is a huge number, which means a small strain will lead to a large stress. In the elongation/compression case, the finite strain can be defined as the relative change of the length \( \delta L/L \), where \( L \) is the length of the spring.

Stretching a body in one direction will usually have the effect of changing its shape in others – typically decreasing it. The strain in \( y \)- and \( z \)-directions caused by the stress \( \sigma_{11} \) can be described as

\[
\varepsilon_{22} = \varepsilon_{33} = -\nu \varepsilon_{11} = -\frac{\nu}{E} \sigma_{11},
\]

where \( \nu \) is also a property of the material and called Poisson’s ratio. It can be mathematically shown \( \nu \in (0, 0.5) \) and usually takes values in the range 0.25 – 0.3.

The strain in any direction will be made up of a combination of the norm strain in that direction and the shear strain in other directions. By the superposition, which holds for
linear elasticity, we have the relation
\[ \varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}), \]
\[ \varepsilon_{22} = \frac{1}{E}(-\nu\sigma_{11} + \sigma_{22} - \nu\sigma_{33}), \]
\[ \varepsilon_{33} = \frac{1}{E}(-\nu\sigma_{11} - \nu\sigma_{22} + \sigma_{33}). \]

For the shear stress and shear strain, one can show

(10) \[ \varepsilon_{ij} = \frac{1}{G}\sigma_{ij} \quad \text{for } i \neq j, \quad \text{where } G = \frac{E}{1 + \nu} \]

is called the modulus of elasticity in shear or the modulus of rigidity.

Exercise 2.1. To derive (10), we consider the effect of applying the normal stress \( \sigma_{33} = \sigma, \sigma_{22} = -\sigma \) and \( \sigma_{11} = -\sigma \). Along planes at 45° the normal stress is zero while the shear stress is \( \sigma \).

In summary, we can write the relation between the strain \( \varepsilon \) and the stress \( \sigma \) in the matrix form

(11) \[
\begin{pmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{12} \\
\varepsilon_{13} \\
\varepsilon_{23}
\end{pmatrix} = \frac{1}{E}
\begin{pmatrix}
1 & -\nu & -\nu & 0 & 0 & 0 \\
-\nu & 1 & -\nu & 0 & 0 & 0 \\
-\nu & -\nu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 + \nu & 0 & 0 \\
0 & 0 & 0 & 0 & 1 + \nu & 0 \\
0 & 0 & 0 & 0 & 0 & 1 + \nu
\end{pmatrix}
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{13} \\
\sigma_{23}
\end{pmatrix}.
\]

We can write in the simple form

(12) \[ A\sigma = \varepsilon(u), \]

where for isotropic elasticity,

(13) \[ A\sigma = \frac{1}{2\mu}\left(\sigma - \frac{\lambda}{2\mu + d\lambda}\text{tr}(\sigma)I\right). \]

Inverting the matrix, we have the relation

(14) \[ \sigma = C\varepsilon \]
where
\[
C = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{pmatrix}
1 - \nu & \nu & 0 & 0 & 0 \\
\nu & 1 - \nu & \nu & 0 & 0 \\
\nu & \nu & 1 - \nu & 0 & 0 \\
0 & 0 & 0 & 1 - 2\nu & 0 \\
0 & 0 & 0 & 0 & 1 - 2\nu \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Comparing (15) with the constitutive equation
\[
\sigma = \lambda \text{trace}(\varepsilon)I + 2\mu \varepsilon,
\]
we obtain the relation
\[
E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)},
\]
and
\[
\lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2(1 + \nu)},
\]
which implies \(\nu \in (0, 0.5)\) and \(\lim_{\nu \to 0.5} \lambda = \infty\). When \(\nu\) is close to 0.5, it will cause difficulties in the numerical approximation.

We have derived the constitutive equation under the assumption: the material is isotropic and the deformation is linear. In general, the constitutive equation can be still described by (14). Mathematically the \(9 \times 9\) matrix \(C\) contains 81 parameters. But \(C\) is a 4-th rank tensor and \(\sigma_{ij} = C_{ijkl}\varepsilon_{kl}\). The symmetry of \(\varepsilon\) and \(\sigma\) reduces the number of independent parameters to 36. For isotropic material, \(C = C'\), where \(C'\) is the matrix in another coordinate obtained by rotation. By choosing special rotations, one can show that \(C\) depends only on two parameters: the pair \((E, \nu)\) or the Lamé constants \((\lambda, \mu)\).

Taking a small volume \(V\), the volume change is
\[
\delta V = \int_{\partial V} \mathbf{u} \cdot \mathbf{n} \, dS = \int_V \text{div} \, \mathbf{u} \, d\mathbf{x} = \text{tr}(\varepsilon).
\]
Therefore the Lamé constant \(\lambda\) describes the stress due to change in density.

3. Tensors

We choose a Cartesian coordinate to describe the deformation. If we change the coordinate, the description will be different. The deformation itself is, however, intrinsic, i.e., independent of the choice of coordinates. Thus different description in different coordinates should be related by some rules. Mathematically speaking, the stress and strain tensors are equivalent class of matrix functions. The equivalent class is defined by changing of coordinates.

3.1. Forces. Recall that \(\Phi : \Omega_R \to \Phi(\Omega_R)\) is the the deformation mapping. Here we use \(\Omega_R\) to indicate it is the reference domain. The body force \(f\) is a vector composed by 3-forms. Therefore
\[
\int_V f(x) \, d\mathbf{x} = \int_{\Omega_R} f_R(x_R) \, dx_R,
\]
together with \(dx = \det(\nabla \Phi) \, dx_R\), its transformation satisfies
\[
f(x) = \det(\nabla \Phi)^{-1} f_R(x_R).
\]
3.2. **Stress Tensor.** Two different Cartesian coordinates can be obtained from each other by a rigid motion. Since the origin of the Cartesian coordinate systems has no bearing on the definition of $\sigma$, we assume that the new coordinate system is obtained by rotating the old one about its origin, i.e., $x' = Qx$, where $Q$ is unitary.

Recall that $\sigma$ is a linear mapping of vectors. In the original coordinate, it is $t = \sigma n$. Applying $Q$ both sides, we get

$$t' = Qt = Q\sigma^{-1}Qn = Q\sigma Q^T n'.$$

Namely

$$\sigma' = Q\sigma Q^T.$$  

We can thus think the stress is a linear mapping between linear spaces (of vectors). The stress matrix is a representation of this mapping in a specific coordinate.

For a given linear operator $T$, the eigenvector $v$ and corresponding eigenvalue $\lambda$ is defined as $Tv = \lambda v$. By the definition of eigenvalue, it depends only on the linear structure not the representation. Therefore the eigenvalues of $\sigma$, and their combination, e.g. $\text{tr}(\sigma) = \lambda_1 + \lambda_2 + \lambda_3$ and $\det(\sigma) = \lambda_1\lambda_2\lambda_3$, are invariant in the change of coordinate.

3.3. **Strain Tensor.** The geometry (length and angle) is also intrinsic, i.e., independent of the choice of the coordinate. Mathematically suppose $x' = Qx$. Then

$$x'T\epsilon'y' = x^T Q^T \epsilon' Q y = x^T \epsilon y.$$

Namely

$$\epsilon' = Q\epsilon Q^T.$$  

Therefore $\text{tr}(\epsilon) = \text{div} u$ is invariant in the change of coordinate.

3.4. **Relation between Trace, Strain and Stress.** Let $\mathbb{M}$ be the linear space of second order tensor ($d \times d$ matrix) and $\mathbb{S} \subset \mathbb{M}$ the subspace of symmetric tensor. We introduce a Frobenius inner product in $\mathbb{M}$ as

$$(A, B)_F = A : B := \text{tr}(B^T A) = \text{tr}(A^T B),$$

which induces the Frobenius norm $\| \cdot \|_F$ of a matrix. For a scalar $p$, we could embed it into $\mathbb{S}$ by $pI_d$ and denote this mapping by

$$\pi : \mathbb{R} \rightarrow \mathbb{S}, \quad \pi(p) = pI_d,$$

and the imagine is denoted by $\mathbb{R}I_d$, where $I_d$ is the identity matrix.

Define $T_R : \mathbb{M} \rightarrow \mathbb{R}I_d$ as the composition $T_R = d^{-1}p \circ \text{tr}$ which is indeed the projection in $F$-product:

$$\quad (T_R \sigma, pI_d)_F = (\sigma, pI_d)_F, \quad \forall p \in \mathbb{R}. $$

The orthogonal complement $(I - T_R)\sigma$ is called the deviation and denoted by $\sigma^D$. We summarize the decomposition as

$$\sigma = (I - T_R)\sigma \oplus T_R\sigma = \sigma - \text{tr}(\sigma)I_d/d + \text{tr}(\sigma)I_d/d.$$  

The differential operators $\epsilon, \text{grad},$ and $\text{div}$ are related by the identity:

$$\text{tr } \epsilon(u) = \text{div } u, \quad \text{div } \pi p = \text{grad } p,$$

which can be verified by direct computation and summarized as the triangle diagram
4. VARIATIONAL FORMS

We define the energy

\[ E(u, \varepsilon, \sigma) = \int_{\Omega} \left( \frac{1}{2} \varepsilon : \sigma - f \cdot u \right) \, dx + \int_{\Gamma_N} g \cdot u \, dx. \]

and consider the optimization problem

\[
\inf_{(u, \varepsilon, \sigma)} E(u, \varepsilon, \sigma)
\]
subject to the relation

\[ \varepsilon = \nabla u, \quad \text{and} \quad \sigma = C\varepsilon = \lambda \text{tr}(\varepsilon)I + 2\mu \varepsilon. \]

Here \( \Gamma_N \) is an open subset of the boundary \( \partial \Omega \) and the complement is denoted by \( \Gamma_D \), i.e., \( \Gamma_D \cup \Gamma_N = \partial \Omega \).

4.1. Displacement Formulation. We eliminate \( \varepsilon \) and \( \sigma \) to get the optimization problem

\[ \inf_{u \in H^1_D(\Omega)} E(u), \]

where \( H^1_D = \{ v \in (H^1(\Omega))^3 : v(x) = 0 \text{ for } x \in \Gamma_D \} \), and

\[
E(u) = \int_{\Omega} \left( \frac{1}{2} \nabla^s u : C\nabla^s u - f \cdot u \right) \, dx + \int_{\Gamma_N} g \cdot u \, dx
\]

\[
= \int_{\Omega} \left( \frac{\lambda}{2} (\text{div} u)^2 + \mu \nabla^s u : \nabla^s u - f \cdot v \right) \, dx + \int_{\Gamma_N} g \cdot u \, dx.
\]

The classic formulation is

\[ -2\mu \text{div} \nabla^s u - \lambda \text{grad} \, \text{div} u = f \quad \text{in } \Omega, \]
\[ u = 0 \quad \text{on } \Gamma_D, \quad \sigma(u) \cdot n = g \quad \text{on } \Gamma_N. \]

The weak formulation is: find \( u \in H^1_D(\Omega) \) such that

\[ (\mathcal{C}\nabla^s u, \nabla^s v) = (f, v) - (g, v)_{\Gamma_N} \quad \text{for all } v \in H^1_D(\Omega). \]

To establish the well posedness of (20), it is crucial to have the following Korn’s inequality

**Theorem 4.1.** There exists a constant \( C(\Omega, \lambda, \mu) \) such that for all \( v \in H^1(\Omega) \)

\[ (\mathcal{C}\nabla^s v, \nabla^s v) + \|v\|^2 \geq C(\Omega, \lambda, \mu)\|v\|_1^2. \]
Proofs of Korn’s inequality are non-trivial for general $H^1$ functions. For pure Dirichlet boundary condition, i.e. $v \in H^1_0(\Omega)$, using the identity
\[ 2\nabla^s v : \nabla^s v - \nabla v : \nabla v = \text{div}[(v \nabla) v - (\text{div} v) v] + (\text{div} v)^2, \]
one can prove that
\[ |v|_1 \leq \sqrt{2} \|\nabla^s v\|. \]
In general, we can use the norm equivalence
\[ \|v\|^2 \approx \|\nabla v\|^2 + \|v\|^2 \quad \text{for all} \quad v \in L^2(\Omega), \]
and the identity
\[ \frac{\partial^2 u}{\partial x_k \partial x_j} = \frac{\partial \varepsilon_{ik}}{\partial x_j} + \frac{\partial \varepsilon_{ij}}{\partial x_k} - \frac{\partial \varepsilon_{jk}}{\partial x_i}, \]
to prove the Korn equality.

**Exercise 4.2.** Prove the identity (23) and use (22) to prove the Korn inequality.

**Exercise 4.3.** Prove the well-posedness of (20) using Korn’s inequality and figure out the dependence of the stability constant on the Lame constants.

### 4.2. Mixed Formulation of Hellinger and Reissner

We keep the displacement and stress $(u, \sigma)$ and eliminate the strain $\varepsilon$ to get the saddle-point problem
\[ \inf_{\sigma} \sup_u \mathcal{E}(u, \sigma) = \inf_{(u, \sigma)} \left( \frac{1}{2} A\sigma : \sigma - f \cdot u \right) \int_\Omega + \int_{\Gamma_N} g \cdot u \, dS, \]
subject to
\[ -\text{div} \sigma = f, \quad \sigma \in H^\text{div}(\Omega, S), \quad \text{div} \sigma = g \quad \text{on} \quad \Gamma_N. \]

Or equivalently
\[ \sigma = C\nabla^s u \in \Omega \quad \sigma \cdot n = g \quad \text{on} \quad \Gamma_N. \]

The displacement can be mathematically treat as the Lagrange multiplier to impose the constraint. Note that $\sigma$ is a symmetric matrix function taking values in $S = \mathbb{R}^{d \times d}_{\text{sym}}$. Differential operators are applied row-wise to a matrix.

The strong formulation is
\[ -\text{div} \sigma = f, \quad \sigma = C\nabla^s u \in \Omega, \]
\[ u = 0 \quad \text{on} \quad \Gamma_D, \quad \sigma \cdot n = g \quad \text{on} \quad \Gamma_N. \]

Define $H_g(\text{div}, \Omega, S) = \{ \tau \in H(\text{div}, \Omega, S), \tau \cdot n = g \quad \text{on} \quad \Gamma_N \}$. The weak formulation is: find $\sigma \in H_g(\text{div}, \Omega, S)$ such that
\[ (A\sigma, \tau) + (u, \text{div} \tau) = 0 \quad \text{for all} \quad \tau \in H_0(\text{div}, \Omega, S) \]
\[ -(\text{div} \sigma, v) = (f, v) \quad \text{for all} \quad v \in L^2(\Omega). \]

When $\Gamma_N = \emptyset$, i.e., $u = 0$ on $\partial \Omega$, the mixed formulation is in the pure Neumann type and solutions are not uniquely. We need to consider the quotient space
\[ \mathring{H}(\text{div}, \Omega) = \{ \tau \in H(\text{div}, \Omega) : \int_\Omega \text{tr}(\tau) \, dx = 0 \}. \]
The constraint comes from, for \( \tau = \varepsilon(v) \),
\[
\int_{\Omega} \operatorname{tr}(\tau) \, d\mathbf{x} = \int_{\Omega} \operatorname{div} \mathbf{v} \, d\mathbf{x} = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \, dS = 0.
\]

Denote the space for stress by \( \Sigma \) equipped with norm \( \| \cdot \|_\Sigma \) and the space for displacement by \( V \) with norm \( \| \cdot \|_V \). Let us introduce the linear operator \( L : \Sigma \times V \to (\Sigma \times V)^* \)
\[
\langle L(\sigma, u), (\tau, v) \rangle := (A\sigma, \tau) - \langle \varepsilon(u), \tau \rangle + \langle \operatorname{div} \sigma, v \rangle.
\]

Define bilinear forms \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \): there exist constants \( c_a, c_b > 0 \) such that
\[
a(\sigma, \tau) \leq c_a \| \sigma \|_\Sigma \| \tau \|_\Sigma, \quad b(\tau, v) \leq c_b \| \tau \|_\Sigma \| v \|_V, \quad \text{for all } \sigma, \tau \in \Sigma, v \in V.
\]

(2) Coercivity of \( a(\cdot, \cdot) \) in the kernel space. There exists a constant \( \alpha > 0 \) such that
\[
a(\sigma, \sigma) \geq \alpha \| \sigma \|_\Sigma^2 \quad \text{for all } \sigma \in \text{ker}(\operatorname{div}),
\]
where \( \text{ker}(\operatorname{div}) = \{ \tau \in \Sigma : b(\tau, v) = 0 \text{ for all } v \in V \} \).

(3) Inf-sup condition of \( b(\cdot, \cdot) \). There exists a constant \( \beta > 0 \) such that
\[
\inf_{v \in V, v \neq 0} \sup_{\tau \in \Sigma, \tau \neq 0} \frac{b(\tau, v)}{\| \tau \|_\Sigma \| v \|_V} \geq \beta.
\]

The continuity condition usually can be easily satisfied by choosing appropriate norm. Inf-sup condition is also relatively easy. The difficulty is the coercivity of \( a(\cdot, \cdot) \).

To illustrate this difficulty, we first use the pair \( \Sigma \times V = L^2(\Omega; \mathbb{S}) \times H^1_0(\Omega; \mathbb{R}^d) \). Now \( \operatorname{div} \sigma \in H^{-1}(\Omega; \mathbb{R}^d) \) is understood as the dual operator, i.e., \( b(\tau, v) = -\langle \varepsilon(v), \tau \rangle \). The inf-sup condition holds trivially by choosing \( \tau = -\varepsilon(v) \) and Korn inequality.

The coercivity of \( a(\cdot, \cdot) \) in \( L^2 \)-norm is in trouble. The \( L^2 \)-inner product for two tensors indeed involves two inner product structures: one is \( \langle \cdot, \cdot \rangle_F \) among tensors and another is the \( L^2 \)-inner product, i.e., \( \int_{\Omega} f \cdot g \, d\mathbf{x} \) of scalar functions. The orthogonality in \( \langle \cdot, \cdot \rangle_F \) will be inherit in \( \langle \cdot, \cdot \rangle \).

**Lemma 4.4.** Let \( T_R \sigma = d^{-1} \operatorname{tr}(\sigma) I_d \). Then
\[
a(\sigma, \tau) = \frac{1}{2\mu} ((I - T_R)\sigma, (I - T_R)\tau) + \frac{1}{d\lambda + 2\mu} (T_R(\sigma), T_R(\tau)),
\]
Proof. Using the formulae of \( A \), c.f. (13), with parameter \( \rho = d\lambda/(d\lambda + 2\mu) \in (0, 1) \), we have
\[
2\mu A\sigma = \sigma - \rho T_R \sigma = (I - T_R)\sigma + (1 - \rho) T_R \sigma.
\]
Recall that \( T_R \) is the projector in \( \langle \cdot, \cdot \rangle_F \) inner product. Using the property of orthogonal projectors, we have
\[
((I - T_R)\sigma, \tau) = ((I - T_R)\sigma, (I - T_R)\tau), \quad (T_R \sigma, \tau) = (T_R \sigma, T_R \tau)
\]
and the identity then follows. \( \square \)

This implies the coercivity on the whole space: for all \( \sigma \in \Sigma 
\]
\[
a(\sigma, \sigma) \geq \min \left\{ \frac{1}{2\mu}, \frac{1}{d\lambda + 2\mu} \right\} \| \sigma \|^2.
\]
The constant $\alpha$, unfortunately, is in the order of $O(1/\lambda)$ as $\lambda \to +\infty$. Namely it is not robust to $\lambda$.

We then consider the pair $\Sigma \times V = \hat{H}(\text{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathcal{V})$. The inf-sup condition is also trivial. Now the coercivity is only needed in the kernel space. It is helpful to use the following triangle diagram connecting the linear elasticity with the inf-sup stability of Stokes equations

$$
\begin{array}{c}
\varepsilon \\
\text{tr} \\
\text{div} \\
\hat{H}(\text{div}, \Omega; \mathbb{S}) \\
\downarrow \\
L^2(\Omega)
\end{array}
$$

FIGURE 3. Linear elasticity and the inf-sup stability of Stokes equations

Lemma 4.5.

(25)  \[ \| T_R \sigma \| ^2 \leq C \left( \| (I - T_R) \sigma \| ^2 + \| \text{div} \sigma \| _{L^2} ^2 \right), \]  for all $\sigma \in \Sigma$.

Proof. Let $p = \text{tr} \sigma$. By the inf-sup stability of Stokes equation, we have

$$
\| p \| \leq C \sup_{u \in H^1_0(\Omega, \mathbb{R}^d)} \frac{\langle \text{div} u, p \rangle}{\| u \| _1}.
$$

Using $\text{div} = \text{tr} \varepsilon$ and $p = \text{tr} \sigma$, we compute

$$
d^{-1}(\text{div} u, p) = d^{-1}(\text{tr} \varepsilon(u), \text{tr} \sigma) = (T_R \varepsilon(u), T_R \sigma) = (\varepsilon(u), T_R \sigma)
= \langle \varepsilon(u), (T_R - I) \sigma \rangle + \langle \varepsilon(u), \sigma \rangle = \langle \varepsilon(u), (T_R - I) \sigma \rangle - \langle u, \text{div} \sigma \rangle.
$$

Then using Cauchy-Schwarz inequality, we get (25). $\square$

We then obtain a robust coercivity of $a(\cdot, \cdot)$ restricted to the null space $\text{ker}(\text{div})$.

Theorem 4.6. There exists a constant independent of $\lambda$ such that

(26)  \[ a(\sigma, \sigma) \geq C \| \sigma \| ^2 \]  for all $\sigma \in \Sigma \cap \text{ker}(\text{div})$.


In applications, when we are mostly interested in computing the stresses, we shall use this mixed formulation.

4.3. Mixed formulation of Hu and Washizu. We keep all variables. The weak formulation is to impose the constitutive equation weakly.

5. Finite Element Methods

- Rigid body motions has no effect on the stress or strain. This invariance must be preserved in the discretization.
- Locking effect of displacement formulation.
- Symmetric stress element in the mixed formulation.
5.1. **Locking and Nearly Incompressible Material.** What is the locking effect: finite element approximation produces significantly smaller displacements than it should.

Why: from the large parameter \( \lambda \) and null space of \( \text{div} \) operator. Recall that the linear elasticity equation using displacement \( u \) is

\[
2\mu(\nabla^s u, \nabla^s v) + \lambda(\text{div} u, \text{div} v) = (f, v),
\]

The Lamé constant \( \lambda \to \infty \) as the Poisson ratio \( \nu \to 0 \). The error of naive finite element approximation is amplified by the constant \( \lambda/\mu \), which is not robust as \( \lambda \to \infty \).

Locking is unavoidable if the finite element space \( V_h \) is not big enough. For example, one can show the locking exists if \( V_h \cap \ker(\text{div}) = \{0\} \) and \( \|\text{div} v_h\| \geq C(h)\|v_h\|_1 \).

When \( \lambda \gg 1 \), the material is called nearly incompressible. The quantity \( \text{div} u \) measures the incompressibility of the material. When \( \lambda \gg 1 \), \( \text{div} u \) should be small. Indeed when the domain of interest is smooth or convex, we have the regularity result

\[
\|u\|_2 + \lambda \|\text{div} u\|_1 \leq C\|f\|.
\]

We can introduce an artificial pressure \( p = \lambda \text{div} u \) and rewrite the equation into a perturbed Stokes equations

\[
2\mu(\nabla^s u_h, \nabla^s v) + (p_h, \text{div} v) = (f, v), \quad \text{for all } v \in H^1_D(\Omega),
\]

\[
(\text{div} u_h, q_h) - \frac{1}{\lambda}(p_h, q) = 0, \quad \text{for all } q \in L_2(\Omega).
\]

Then we can use stable finite element methods developed for Stokes equations. For example, the finite element space for the displacement (now is the velocity in Stokes problem) should be fine enough and spaces \( (V_h, P_h) \) should satisfy the inf-sup condition.

\[
2\mu(\nabla^s u_h, \nabla^s v_h) + (p_h, \text{div} v_h) = (f, v_h), \quad \text{for all } v_h \in V_h \subset H^1_D(\Omega),
\]

\[
(\text{div} u_h, q_h) - \frac{1}{\lambda}(p_h, q_h) = 0, \quad \text{for all } q_h \in P_h \subset L_2(\Omega).
\]

In general, \( \text{div} V_h \not\subset P_h \). We define \( Q_h \) as the \( L^2 \) projection to \( P_h \). Taking \( q_h = Q_h \text{div} v_h \) and addition these two equations, we obtain a modified formulation

\[
2\mu(\nabla^s u_h, \nabla^s v_h) + \lambda(Q_h \text{div} u_h, Q_h \text{div} v_h) = (f, v_h).
\]

The inner product of \( L^2 \) projection can be thought of as a selective reduced integration.

5.2. **Symmetric stress and discontinuous displacement.** The trick to recover the coercivity \((25)\) can be applied to the discretization provided that \( \ker(B_h) \subset \ker(B) \) which is equivalent to the requirement \( \text{div}(\Sigma_h) \subset U_h \). With this consideration the natural finite element spaces would be \( P_{k+1}(T_h; S) - P_k^{-1}(T_h; V) \) for \( k \geq 0 \). The discrete inf-sup condition is now more difficult to verify.

Hu and Zhang’s work

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**REFERENCES**