The Cauchy-Schwarz Inequality

The Cauchy-Schwarz Inequality is one of the most important inequalities in mathematics. It constantly appears in numerous branches of mathematics and it is an invaluable tool for problem solving. The Cauchy-Schwarz inequality is as follows:

Cauchy-Schwarz Inequality

Let $a_1, ..., a_n$ and $b_1, ..., b_n$ be real numbers. Then

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$$

For this packet, assume all numbers are real unless stated otherwise. Let's do a few examples to convince you that this inequality is true.

Example 0.1 *Let* $(a_1, a_2, a_3) = (3, 4, 4)$ *and* $(b_1, b_2, b_3) = (0, 2, 3)$. *Then the left hand side of the Cauchy-Schwarz inequality is*

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 = (3 \cdot 0 + 4 \cdot 2 + 4 \cdot 3)^2 = 20^2 = 400$$

while the right hand side of the Cauchy-Schwarz inequality is

$$\begin{aligned}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) &= (3^2 + 4^2 + 4^2)(0^2 + 2^2 + 3^2) \\ &= (9 + 16 + 16)(0 + 4 + 9) \\ &= (41)(13) \\ &= 533.\end{aligned}$$

Since 400 < 533,

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

which verifies that the Cauchy-Schwarz inequality holds.

Problem 0.2 *Verify that the Cauchy-Schwarz inequality holds for the following numbers:*

1. $(a_1, a_2) = (-3, 4)$ *and* $(b_1, b_2) = (3, 2)$
2. $a_1 = 3, b_1 = 4$
3. $(a_1, a_2) = (1, 1)$, $(b_1, b_2) = (3, 3)$

4. $(a_1, a_2, a_3) = (1, 2, 3)$, $(b_1, b_2, b_3) = (-2, -4, -6)$

5. $(a_1, a_2, a_3) = (2, 0, 0)$, $(b_1, b_2, b_3) = (2, 1, 0)$

Record your answer in the table below. The first line of the table has been filled in as an example.

$(a_1,\ldots,a_n)$	$(b_1,\ldots,b_n)$	$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$	$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$
(3, 4, 4)	(0, 2, 3)	400	533
(-3, 4)	(3, 2)		
3	4		
(1, 1)	(3,3)		
(1, 2, 3)	(-2, -4, -6)		
(2, 0, 0)	(2,1,0)		

Figure 1: Table 1

You have done some computations to get used to the Cauchy-Schwarz inequality. Now try some of these problems.

Problem 0.3 *When $n = 1$, show that the Cauchy-Schwarz inequality is true; that is, show that if a_1 and b_1 are any real numbers, then*

$$(a_1b_1)^2 \leq (a_1^2)(b_1^2)$$

Problem 0.4 *When $n = 2$, show that the Cauchy-Schwarz inequality is true; that is, show that if a_1, a_2 and b_1, b_2 are any real numbers, then*

$$(a_1b_1 + a_2b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)$$

(Hint: Expand out both sides of the inequality, then simplify. You may need to use the inequality $(x - y)^2 \geq 0$.)

Problem 0.5 *Use the Cauchy-Schwarz inequality to prove that*

$$1^2 + 2^2 + \cdots + n^2 \geq \frac{(1 + 2 + \cdots + n)^2}{n}$$

(Hint: Pay attention to the n in the denominator. $n = 1 + \cdots + 1$ where there are n 1s.)

Look back at Table 1. Notice that sometimes the Cauchy-Schwarz inequality is an equality. That is, for some a_k's and b_k's we have

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 = (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$$

In Table 1, you showed that this is true when

$$a_1 = 3 \text{ and } b_1 = 4$$

$$(a_1, a_2) = (1, 1) \text{ and } (b_1, b_2) = (3, 3)$$
$$(a_1, a_2, a_3) = (1, 2, 3) \text{ and } (b_1, b_2, b_3) = (-2, -4, -6)$$

Notice that in each of these three cases the right hand side is a multiple of the left hand side:

$$4 = \frac{4}{3} \cdot 3$$
$$(3, 3) = (3 \cdot 1, 3 \cdot 1)$$
$$(-2, -4, -6) = (-2 \cdot 1, -2 \cdot 2, -2 \cdot 3)$$

With this in mind, solve the following problem.

Problem 0.6 *Let r, $a_1, ..., a_n$ and $b_1, ..., b_n$ be numbers so that for every k with $1 \leq k \leq n$, $b_k = r \cdot a_k$. Show that the Cauchy-Schwarz inequality is actually an equality; that is under these conditions*

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 = (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$$

We have done problems which test your ability to use the Cauchy-Schwarz. Now we will prove that the inequality is true. First we need to briefly review some facts about polynomials.

Definition 0.7 *A polynomial p(x) is said to* ***quadratic*** *or* ***degree 2*** *if it is of the form $p(x) = ax^2 + bx + c$ If x_0 is a number so that $p(x_0) = 0$, then x_0 is called a* ***root*** *of the polynomial $p(x)$.*

Note that in this section we will only consider polynomials with real coefficients and only consider roots that are real numbers. If this terminology is unfamiliar to you, then just ignore this paragraph

Problem 0.8 *Give an example of a quadratic polynomial with no roots. Give an example of a quadratic polynomial with exactly one root. Give an example of a quadratic polynomial with 2 distinct roots.*

Problem 0.9 *Show that if a quadratic polynomial* $p(x)$ *has 2 distinct roots, then part of the graph of the polynomial lies below the* x*-axis; that is, there is a real number* x_0 *so that* $p(x_0) < 0$. *(Hint: If a polynomial has two distinct roots, how can it be factored?)*

Problem 0.10 *Conclude that if a quadratic polynomial* $q(x)$ *satisfies* $q(x) \geq 0$ *for every* x*, then* $q(x)$ *can have at most one root.*

Recall from your algebra class that if $p(x) = ax^2 + bx + c$, then $p(x_0) = 0$ if and only if

$$x_0 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The expression underneath the square root sign, $b^2 - 4ac$, is called the **discriminant** of the polynomial $p(x)$.

Problem 0.11 *Conclude that if* $q(x)$ *is a quadratic polynomial with* $q(x) \geq 0$ *for every* x*, then the discriminant is less than or equal to 0.*

Problem 0.12 *On the other hand, show that if the discriminant is 0, then $q(x)$ has one root. Similarly, show that if the discriminant is negative, then $q(x)$ has no roots.*

Now we will return to proving the Cauchy-Schwarz inequality. Let $a_1, ..., a_n$ and $b_1, ..., b_n$ be numbers and for the remainder of this section let $h(x) = (a_1 x + b_1)^2 + \cdots + (a_n x + b_n)^2$.

Problem 0.13 *Is $h(x) \geq 0$ for every number x? Rewrite $h(x)$ so that it is of the form $c_2 x^2 + c_1 x + c_0$. How many roots can $h(x)$ have? (Hint: For the last part use problem 0.10.*

Problem 0.14 *Solve the equation $h(x) = 0$ using the quadratic equation. Write down an inequality for the discriminant. Conclude that the Cauchy-Schwarz inequality is true.*

You previously showed that if there is an r so that $(b_1, ..., b_n) = (r \cdot a_1, ..., r \cdot a_n)$, then the Cauchy-Schwarz inequality becomes an equality. Now, let's show almost the reverse statement. Let's show that if the Cauchy-Schwarz inequality is actually an equality

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 = (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2),$$

and $a_k, b_k \neq 0$ for all k, then there is an r so that $(b_1, ..., b_n) = (r \cdot a_1, ..., r \cdot a_n)$.

Problem 0.15 *Recall that* $h(x) = (a_1x + b_1)^2 + \cdots + (a_nx + b_n)^2$. *Suppose also that* $a_k, b_k \neq 0$ *for all* k. *If* $h(x_0) = 0$, *then for any* k *with* $1 \leq k \leq n$, *what is* $(a_kx_0 + b_k)$ *equal to? In this case* x_0 *must equal* n *different expressions at once. Write down the* n *things that* x_0 *is equal to.*

Problem 0.16 *Suppose once again that in* $h(x)$ *for all* k, $a_k, b_k \neq 0$. *If the Cauchy-Schwarz inequality is actually an equality for these* a_k*'s and* b_k*'s, then why must there exist an* x_0 *so that* $h(x_0) = 0$*? Conclude that there is a number* r *so that*

$$(b_1, ..., b_n) = (r \cdot a_1, ..., r \cdot a_n).$$

0.1 More Problems on the Cauchy-Schwarz Inequality

Problem 0.17 *Consider the function* $f(x) = \frac{(x+k)^2}{x^2+1}$ *where* k *is a positive whole number. Show that* $f(x) \leq k^2 + 1$.

Problem 0.18 *For positive reals* $a_1, a_2, ..., a_n$ *and* $b_1, b_2, ..., b_n$ *prove that*

$$\frac{x_1^2}{y_1} + \cdots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + \cdots x_n)^2}{y_1 + \cdots + y_n}$$

(Hint: In the Cauchy-Schwarz inequality, let $a_k = \frac{x_k}{\sqrt{y_k}}, b_k = \sqrt{y_k}$

Problem 0.19 *Suppose that* $p_j \geq 0$ *for all* $j = 1, 2, ..., n$ *and that* $p_1 + p_2 + \cdots + p_n = 1$. *Show that if* a_j *and* b_j *are nonnegative real numbers that satisfy the bound* $1 \leq a_j b_j$ *for all* $j = 1, 2, ..., n$ *then one also has*

$$1 \leq \left\{p_1 a_1 + \cdots + p_n a_n\right\}\left\{p_1 b_1 + \cdots + p_n b_n\right\}$$

Problem 0.20 *Show that for all positive x, y, z one has*

$$\left(\frac{x+y}{x+y+z}\right)^{1/2} + \left(\frac{x+z}{x+y+z}\right)^{1/2} + \left(\frac{y+z}{x+y+z}\right)^{1/2} \le 6^{1/2}$$

Problem 0.21 *Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be positive numbers so that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Show that*

$$\frac{a_1^2}{a_1+b_1} + \cdots + \frac{a_n^2}{a_n+b_n} \ge \frac{a_1 + \cdots + a_n}{2}.$$

Problem 0.22 *Suppose that* $p_k > 0$ *for* $1 \leq k \leq n$ *and* $p_1 + p_2 + \cdots + p_n = 1$*. Show that one has the bound*

$$\sum_{k=1}^{n} (p_k + \frac{1}{p_k})^2 \geq n^3 + 2n + 1/n$$

where

$$\sum_{k=1}^{n} (p_k + \frac{1}{p_k})^2 = (p_1 + \frac{1}{p_1})^2 + \cdots + (p_n + \frac{1}{p_n})^2.$$

0.2 Lagrange's Identity

The next series of problems will lead you through a derivation of Lagrange's Identity. You will need to be familiar with Σ notation in order to complete these problems.

Lagrange's Identity

Let $a_1, ..., a_n$ and $b_1, ..., b_n$ be real numbers. Then

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_i b_j - a_j b_i\right)^2$$

Problem 0.23 *If one did not know the Cauchy Schwarz inequality, but knew Lagrange's identity, then how could one derive the Cauchy-Schwarz inequality. (Hint: Write the Cauchy-Schwarz inequality in Σ notation, then compare that expression to the one written above.*

In Σ notation, as you showed in the previous problem, the Cauchy-Schwarz inequality says that

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Loosely speaking, Lagrange's identity says that the left hand side in the Cauchy-Schwarz inequality is off from the right hand side of the Cauchy-Schwarz inequality by the error term $\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2$.

Problem 0.24 *Prove Lagrange's identity.*