TRANSFINITE RECURSION AND COMPUTATION IN THE ITERATIVE CONCEPTION OF SET

BENJAMIN RIN
UNIVERSITY OF CALIFORNIA, IRVINE

ABSTRACT. Transfinite recursion is an essential component of set theory. In this paper, we seek intrinsically justified reasons for believing in recursion and the notions of higher computation that surround it. In doing this, we consider several kinds of recursion principles and prove results concerning their relation to one another. We then consider philosophical motivations for these formal principles coming from the idea that computational notions lie at the core of our conception of set. This is significant because, while the iterative conception of set has been widely recognized as insufficient to establish Replacement and recursion, its supplementation by considerations pertaining to algorithms suggests a new and philosophically well-motivated reason to believe in such principles.

1. Introduction

It has been often noted that the notion of infinity that we find in pre-19th century sources is vastly different from the notion that we inherit from Cantor and that is tacit in the axioms of ZFC set theory. For instance, pre-Cantorian notions tend to accept the principle that a proper subset of the natural numbers should be smaller than the natural numbers themselves, a principle which the Cantorian theory of cardinality happily violates. In contrast to the pre-Cantorian conception, a distinctive feature of the Cantorian conception of the infinite is that we can apply induction to ordinal and cardinal numbers. That said, induction was not the only feature of the natural numbers which was transferred to the infinite: for, in Cantor's works and practice, we find that the notion of recursion was simultaneously transferred to the infinite.

While much in the extant literature on the philosophy of set theory has attended to the idea of a well-order inherent in the Cantorian conception of the infinite, less attention has been paid to the notion of recursion, and this paper seeks to fill this gap in the literature. To be sure, part of this lack of attention can be traced to the fact that the iterative conception of set is sometimes thought not to justify reflection and Replacement, either of which may be used to give the standard proof of transfinite recursion. Our guiding questions here are thus: what notions of recursion and computation are available for Cantor's infinite collections, and of what philosophical principles about the notion of set can we avail ourselves to validate these principles?

Part of the difficulty in answering these questions resides in the comparative opacity of the conception of computation even in the ordinary setting of the natural numbers. There we have a confluence of different models (Turing programs, general recursive functions, etc.), but this confluence does not seemingly deliver an axiomatic conception of computation in the same way that the Peano axioms deliver an axiomatic conception of number. Despite this, it is suggested in this paper that some light can be shed on these conceptual questions by asking after the pre-theoretic ideas implicit in recent work by Hamkins, Koepke, and Welch on infinite time and tape Turing machines. So, in this paper, we proceed by contrasting these pre-theoretic ideas to those behind Kreisel and Sacks' work on higher recursion theory and some related ideas of Jensen and Karp on transfinite analogues of primitive recursion.

Section 2 begins with some historical examples of uses of transfinite recursion. While the standard proof of the theorem of transfinite recursion using Replacement is of course correct, it is natural to ask after reasons for believing in transfinite recursion that are more strongly rooted in our foundational notions of set and ordinal. Next we consider Kanamori's history of the Axiom of Replacement and suggest that it does not provide a convincing case for the Axiom to someone who is interested in strong intrinsic reasons for accepting such principles. In section 3, we define some axioms to serve as a basic initial attempt at formalizing principles to potentially ground recursion. Realizing how closely these relate to computational notions, we look in section 4 at the possibility of grounding recursion on metacomputational notions falling from Koepke's so-called ordinal Turing machines (OTMs). It is argued that such machines present good candidates for answering a challenge once cast by Kreisel concerning the possibility of a notion of higher computation corresponding to his (and later Sacks') notion of higher recursion. We conclude by considering the principle, "Algorithms determine functions," and offer some initial attempts to formalize it. The intent is to advance this as a foundational principle intrinsic to our fundamental conception of set.

Ultimately, the aim of this paper will be less to provide definitive answers to the guiding questions, and more to provide a reasonable starting point that draws attention to an area which until now has received less notice than it deserves. The attempts here to formalize the intuition that algorithms determine functions will be seen to encounter challenges whose solutions are non-trivial and warrant further research. Still, it is hoped that the philosophical intuition proposed here will spur further consideration that eventually leads to the right kind of formal theory.

2. The Practice and Justification of Transfinite Recursion

The first historical use of transfinite recursion is due to Cantor, in the proof of what is now called the Cantor-Bendixson Theorem.[1] The theorem states that every closed uncountable set X of real numbers contains a perfect subset. A *perfect set* is a set containing no isolated points.[2] Cantor's proof of the theorem uses a procedure that we now recognize as transfinite recursion. First, he begins with the original set X, then removes all the isolated points from it. Naïvely, it would seem that the task is finished, but the resulting set—call it X_1—may well have points in it that were not isolated in X but have now become isolated thanks to the removal of some of its surrounding points. So Cantor removes all the now-isolated points from X_1, resulting in a new set X_2, and so on.

Cantor's insight was that this process may be iterated recursively (indeed, transfinitely recursively), and if it is, it will eventually reach a stage X_α when no isolated points remain to be removed. When this happens, the process can stop. The recursive function he defined that drives the proof is formally given by: $X_0 = X$, $X_{\alpha+1} = X_\alpha \setminus I(X_\alpha)$ and $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$ for $X \subseteq \mathbb{R}$, where $I(Y)$ is the set of isolated points in Y. One can argue that the perfect set theorem marked the birth of modern (Cantorian) set theory. For it was here that transfinite ordinals were first used, in order to define a recursive process that does eventually halt but not at any finite stage.

Another historical example of transfinite recursion from the same time period is in the definitions of the basic operations of ordinal arithmetic.[3] Ordinal addition is defined by transfinite iteration of the "successor" operation. Ordinal multiplication is defined by transfinite iteration of the addition operation. Ordinal exponentiation is defined by transfinite iteration of multiplication. These definitions parallel and generalize their finitary counterparts (their restrictions to finite numbers giving exactly the recursive definitions of $+$, $\times$ and exp in Peano Arithmetic). What separates them is of course their behavior at the infinite limit step.

A more recent example of transfinite recursion, familiar to many philosophers, is seen in Kripke's iterative definition of truth (c.f. [Hor11, Ch. 9]). Kripke's truth predicate famously uses a recursive procedure to evaluate sentences like, "It is true that it is true that it is false that it is true that snow is white," from the inside on out. The recursion necessarily caps at some countably infinite ordinal stage, proving that a three-valued language as Kripke defines it can, unlike Tarski's, contain its own truth predicate. Problem sentences like the Liar do not receive a true or false value at any stage.

[1] For a modern treatment, see [Jec03, p. 40]. This result is sometimes called the perfect set theorem.

[2] A real number within a set X is *isolated in* X if none of its neighbors are in X—that is, if there exists an open interval of reals containing it such that no other numbers in the interval are in X.

[3] See, e.g., [Jec03, p. 23].

As a final, and rather iconic, couple of examples of transfinite recursion, we have the definitions of the stages of V, the Von Neumann universe of all sets, and of L, the Gödel universe of constructible sets. Beginning with the empty set $\emptyset = V_0 = L_0$, we recursively define $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ and $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$, where $\mathcal{P}$ is the power set operation and $\mathrm{Def}(Y)$ is the set of definable sets of Y.[4] At limit stages we define $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ and $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$.[5]

The examples of definitions above share the common feature that each depends on applying an iterated procedure that takes the output of one iteration as the input for the next. This results in a function whose existence and well-definedness are guaranteed by the theorem of transfinite recursion (TR). The theorem TR, in turn, is standardly justified in ZFC by a proof reliant on the Axiom of Replacement. Replacement says that the range of a function whose domain is a set must also be an existing set.[6] Its role in the proof of TR is in helping to show inductively that for all ordinals δ, there exists an approximation of the desired recursive function accurate to the first δ output values. In brief, this is made to work because Replacement asserts the existence of the range of a function from domain δ to a set of smaller approximations (each $\delta' < \delta$ maps to an approximation accurate to δ' many values), whose union is then the desired δ-approximation.[7]

However, though Replacement might be a good reason to think that recursive definitions are justified, it is natural to ask: is this the reason why we in fact think that they are justified? Certainly Cantor did not justify his argument this way. Historically, the Axiom of Replacement was not introduced until 1922, and it was not until Von Neumann later made the connection between Replacement and recursion that the two were formally associated.[8] As a matter of actual practice, definitions by recursion require no understanding of or reference to the Axiom of Replacement at all. A mathematician working in topology can perfectly well understand the definition of Cantor's function without this formal grounding. Similarly, a philosopher working on formal theories of truth with little background in set theory can readily understand Kripke's construction without Replacement. Replacement is in practice quite divorced from transfinite recursion, despite the formal entailment.

Of course, to count as being the proper justification for something, an axiom or other principle need not be constantly borne in mind during its practical application. No one questions Cauchy sequences or Dedekind cuts as being legitimate justifications for the existence of real numbers by objecting that earlier mathematicians did not think about them in practice. But nonetheless, where multiple justifications are present, it is natural to think that some are better explanations than others, and that some enhance our understanding better than others. To continue with the analogy with the real numbers, it seems that part of the success of Cauchy sequences (or Dedekind cuts) is that $\{3, 3.1, 3.14, 3.141, 3.1415, \dots\}$ usefully *tells us something* about π. A present-day student—or a 19th-century analyst—who encounters Cauchy sequences would not merely gain a more rigorous knowledge of the existence of π in this way, but would learn something about the nature of π, namely that it is a limit of a certain rational approximation. The properties of such rational approximations can then

[4] I.e., $\mathrm{Def}(Y) = \Big\{\{y \mid y \in Y \text{ and } (Y, \in) \models \Phi(y, z_1, \dots, z_n)\} \;\Big|\; \Phi \text{ is a first-order formula and } z_1, \dots, z_n \in Y.\Big\}$

[5] See, e.g., [Jec03, p. 175].

[6] There are other equivalent formulations, and in first-order set theory the axiom is standardly represented as an infinite schema.

[7] See [Kun80, p. 25] or [Jec03, p. 22].

[8] See [Kan12].

be studied in numerous contexts, like that of transcendental number theory. It is expected, then, that a study of different reasons to believe in transfinite recursion will show that some of these reasons would enhance our understanding of the underlying concept of set, while others would merely justify definitions by transfinite recursion.

To make this point clearer, it is important to become more precise about just what is meant here by 'justifying a definition'. After all, stipulative definitions usually need no justification. So what could this possibly mean?

One (minimal) answer is that it means being sure that our definition actually succeeds at *being a definition*. That is, we want to be sure that there exists a function satisfying the description in the definition (and be sure that there is exactly one such function). Consider the following recursive instruction for computing $f(n)$:

$$f(n) = \begin{cases} \frac{5}{12}, & \text{if } n = 0 \\ \frac{1}{f(n-1)} - 2, & \text{if } n > 0 \end{cases}$$

The equation here offers a recursive description of what looks to be an unobjectionable mathematical function with domain the natural numbers and with outputs in the rationals. To keep the example simple, only regular (finitary) recursion is used, with no transfinite limit step. The recursive process here begins with $f(0) = \frac{5}{12}$. Subsequent values are calculated from previous answers such that $f(1) = \frac{1}{\frac{5}{12}} - 2 = \frac{2}{5}$, from which it follows that $f(2) = \frac{1}{2}$, and thus $f(3) = 0$. Here, though, we encounter a problem: we see straightforwardly that $f(4)$ has an undefined value. The equation above, which initially seemed to define a recursive function, in fact defines nothing. Based on this example, at the bare minimum we should require that justifying a recursive definition means making sure this type of thing does not happen. In short: definitions must define!

But it may be that one could ask for more than this. One might want not only a guarantee that there is a function satisfying the given description, but also for the function to be able to be "read off" the description in some uniform way. One could want, in other words, for there to be a single way to take descriptions of this kind and understand from them the behavior of the function described.

As an even stronger demand, one could also require that the definition be the 'right' one based on considerations arising within the most general relevant areas of mathematical practice. For example, consider how we define a prime number. In [Tap08], Tappenden contrasts two competing definitions. The first is that $a \neq 1$ is prime if its factors are precisely 1 and a. The second is that $a \neq 1$ is prime if it divides b or c whenever it divides bc. These definitions are equivalent in $\mathbb{N}$—they have the same extension $\{2, 3, 5, 7, \ldots\}$—but are not equivalent in alternate or generalized contexts (e.g., in certain instances of modular arithmetic). It turns out that in these other contexts, the first definition fails to entail key results. On this, Tappenden writes:

> Once the two options are recognized, we need to say which will be canonical. [...] The word *prime* is given the *second* definition. [...] The reason for counting the second definition as the proper one is straightforward: The most significant facts about prime numbers turn out to depend on it. [...] The original definition of prime number fixes a set $\{2, 3, 5, 7, \ldots\}$ with interesting properties. In the original domain $\mathbb{N}$ it can be picked out by either definition.

> The new definition is the more important, explanatory property, so it is the natural one. [Tap08, 268]

The justification of the second definition over the first connects strongly with the definition's usefulness and explanatory power in general mathematical practice, e.g., in ring theory. In contrast, the first serves well only in its native setting $\mathbb{N}$. Along these lines, a strong general criterion for justifying definitions could require that a justification for transfinite recursion ensure the definition captures the most general relevant properties of the function defined.

Regardless of what notion of justification is at play here, it is fair to ask: why should one feel the need to justify recursions in the first place? An answer that one could be tempted to give is that we need recursion for various results.[9] However, it is not trivial to formulate clearly what is meant by 'needing' transfinite recursion to do something. What one might have in mind in saying TR is 'needed' for a given result may be that one could choose an appropriate background set theory without it, say ZFC minus Replacement (ZC), and show that it follows formally from the given result. But it's wrong to think that this would be the only type of reason we should give in order to motivate the project of justifying transfinite recursion. If it turned out somehow that we did not need transfinite recursion for anything, even so, the question of its justification is of interest. The goal we have is to pinpoint what good reasons we have for believing in it, not whether it is the only route to this or that result.[10]

One kind of reason that can be offered has to do with to our conception of the ordinals. To the extent that the class Ω of ordinals is thought of as generalizing the notion of natural number, the possibility of performing recursive procedures on them should be viewed as an inherent part of what they are. Just as it is difficult to picture what the naturals $\mathbb{N}$ would 'look like' if it were impossible to perform recursion upon them, it should be equally difficult and bizarre to picture what the ordinals would look like if it were impossible to perform (transfinite) recursion upon *them*. Ordinal numbers *just are* the type of thing that one can use for performing transfinite recursion.

It is well known that the theory ZC can be modeled by the structure $(V_{\omega+\omega}, \in)$. Thus one might picture the collection Ω^- of ordinals-if-recursion-upon-them-were-impossible as simply the class $\omega + \omega$—now a proper class. But I hardly think that anyone believes this to be what the class of all ordinals truly looks like. Later in this paper, it will be seen that there exist recursion principles weaker than TR that are still able to surmount the $\omega + \omega$ barrier. But to eschew recursion above the finite altogether is to invite a very limited picture of the class Ω.

---

[9] Returning to the motivating examples earlier in this section, one may notice that there is actually a certain asymmetry between the Cantor-Bendixson example and the others. In the Cantor-Bendixson case, we are going "down" and not "up", in the sense that the sets being defined at each stage are smaller than those at previous stages. In contrast, the definitions of V_α and L_α, as well as the operations of transfinite arithmetic, result in objects that are unboundedly larger or higher ranked than their predecessors in the recursive construction. One may want to distinguish between these two types of cases, which arguably warrant different treatment. In 1922, the same year that Replacement was introduced, Kuratowski [Kur22] actually gave a general procedure for how to do proofs with "downward" examples like the Cantor-Bendixson function without using recursion (see [Hal84, p. 253*ff.*]). This method certainly does not work for cases like V_α.

[10] In section 3, we prove some results showing that Replacement is actually provable by route of certain types of recursion and other principles. This, of course, is useful. But the point is that even if we did not have results like these, what we are ultimately interested in are reasons for believing in such principles. These we talk about in sections 4 and 5.

Having said that, the smallness of the model $(V_{\omega+\omega}, \in)$ has historically been used to motivate the Axiom of Replacement. As Kanamori recounts in [Kan12], Fraenkel and Skolem pointed out in the '20s to Zermelo that the theory Z, with or without Choice, does not entail the existence of certain sets, such as $\mathcal{P}^\omega(\omega) =_{\text{df}} \{\omega, \mathcal{P}(\omega), \mathcal{P}(\mathcal{P}(\omega)), \ldots\}$. This set is formed, in modern terms, at stage $V_{\omega+\omega}$. In a letter to Fraenkel, partly reproduced in [Kan12], Zermelo agreed that since Z cannot prove the existence of such sets, it cannot by itself be considered a complete theory of sets. But with Replacement, it becomes possible to construct the ordinal $\omega + \omega$ as a set using the function $n \mapsto \omega + n$ for $n \in \omega$; since the domain ω is a set, so must be its range $(\omega + \omega) \setminus \omega$, the union of which is $\omega + \omega$. One constructs the set $\mathcal{P}^\omega(\omega)$ from the function $n \mapsto \mathcal{P}^n(\omega)$ similarly.

However, it has since been recognized that while Replacement solves the $\omega+\omega$ problem, it is not well motivated by the iterative conception of set. For the iterative conception has it that sets are formed at "stages" out of the elements formed in previous stages. Beginning with the empty set, we transfinitely continue forming subsets of a given stage and collecting these subsets together to comprise the next stage of our universe of sets. But in [Boo71], [Boo89], and [Pot04], we see that certain natural attempts at formalizing the iterative conception are in fact satisfiable by models like $(V_{\omega+\omega}, \in)$. Thus the iterative conception, at least under the best current attempts at formalization, fails to justify Replacement. Relatedly, as Martin has expressed: "There are certainly unclear points about this iterative notion of set. What is meant by 'subset' of $[V_\alpha]$? How long does the transfinite process continue?" [Mar70, p. 112]. The latter question points at the underdetermination of the size of the universe by a conception that excludes Replacement (though this type of problem persists even in full ZFC when considering large cardinals).

Given that the iterative conception does not justify Replacement, some might call this a strike against Replacement. Kanamori's attitude seems to be: so much the worse for the iterative conception. Or rather, so much the worse for using it as the sole principle from which to generate a mature set theory.

Kanamori defends his view in [Kan12], offering a history of the Axiom of Replacement with the specified intention of arguing for and highlighting its significance—even centrality—among the axioms of set theory. He aims to push back against the "hesitation and even skepticism [that] have been voiced about the importance and even the need for Replacement" by, e.g., Potter, Boolos, and others. Within his historical account, he cites numerous results that depend on Replacement, in the sense that they are not provable in ZC—for instance, Borel Determinacy, the existence of the ordinal $\omega+\omega$, and of course the theorem of transfinite recursion. In doing this, he compares ZC with ZFC, without attention to other theories that might entail these results.

The main contributions of Replacement, on Kanamori's account, are "functional substitution, recursion, and indifference to identification." The first and last of these draw on the fact that Replacement lets us literally *replace* elements of one set with other existing objects one for one and still be left with a set, allowing us familiar constructions like $\{t_i \mid i \in I\}$ given some specified correlation of elements t_i to members i of an index set I. Thus Replacement allows for a sort of structuralist-style view of mathematical objects such as ordered pairs, real numbers, ordinals, and so forth:

> What Replacement does is to allow for articulations that these representations are not necessary choices and to mediate generally among possible choices.

> Replacement is a corrective for the other axioms, which posit specific sets and subsets, by allowing for a fluid extensionalism. The deepest subtlety here is also on the surface, that through functional correlation one can shift between tokens (instances, representatives) and types (extensions, classes), and thereby shift the ground itself for what the types are. ([Kan12, p. 47])

Some remarks on this are in order. First, the flexibility offered by Replacement is indeed very powerful. To be indifferent among alternative ways to define real numbers, ordinals, and so on, is mathematically quite convenient. But one is much more likely to find this option necessary or even desirable if one antecedently has structuralist leanings. And in the context of set theory, there is hardly any structure at all to sets *in general*, in contrast to, say, groups, well-orderings, or natural number systems. The only structural component in a universe of sets is its relation of elementhood, $\in$. An "isomorphism" between two sets is nothing more than a bijection. Thus a powerful principle like Replacement provides for the existence of sets based solely on their size, the only structural feature that sets in general share. The argument for Replacement based on indifference to identification therefore offers little more than what is offered by a "limitation of size" principle.[11]

But this sort of principle brings with it known difficulties. As discussed in, e.g., [Boo89] and [Hal84, Ch. 5]), there is a tension between limitation of size as a principle of set existence and the iterative conception of set. On the typical characterization, limitation of size says that sets come from classes (or properties) that aren't "too big." If a property is satisfied by a class of elements that is too big, then the elements are just too numerous to be consistently collected into a set. For if they were so collected, we would fall to the paradoxes of naïve set theory. This principle would be a reasonable grounding for Replacement, which the iterative conception fails to justify alone. However, limitation of size is difficult to reconcile with the Axiom of Power Set, as $\mathcal{P}(x)$ can grow enormously large with respect to x, even unpredictably so (see [Hal84, p. 205-208]).[12]

In connection with limitation of size, one might also try to defend Replacement based on the naïve picture, briefly noted earlier, that one simply ought to be able to take an existing set and exchange (or *replace*) its elements one for one with other members of the universe without sacrificing its set-hood. But ultimately, this amounts just to a restatement of what the Axiom of Replacement says. While similar intuitions have been used before to motivate Separation, Separation is weaker than Replacement, and so it is not obvious why a reason for believing that should thereby be a reason to believe in the truth of Replacement.

In contrast to all these attempts, Kanamori's approach is to push aside the issue of needing a single foundational, pre-theoretic idea behind *Set* that would justify both Power Set and Replacement intrinsically. Instead, he points to Replacement's vast usefulness in practice and accepts these two competing principles together on extrinsic grounds. But for one who

[11] See also [Pot04, p. 231].

[12] An important distinction to keep in mind here is between limitation of size and the related but alternative principle Hallett has called Cantorian finitism. Roughly, Cantorian finitism is the idea that infinite sets should be treated as much as possible in the same way that we treat finite sets. One of the tenets is that any set, even an infinite one, should be conceived as a unit. Another one is extensionalism: sets (even infinite ones) consist precisely of their members. (For more on this, see [Hal84, p. 7-9 and 32-40].) With respect to the present discussion, Cantorian finitism can justify the Axiom of Power Set on the grounds that since finite sets always have power sets, so should infinite sets. This sort of argument is independent of arguments based on limitation of size, and so should not be viewed as providing reason for them.

still asks for a single pre-theoretic idea, Kanamori's philosophical argument—the quality of his historical account notwithstanding—does not offer much that is new. One is left in the same position as when, in Potter's words, "Even von Neumann himself admitted that the axiom of replacement, just regarded as an axiom without regard to its useful consequences, 'goes a bit too far'" [Pot93, 182].

If one were to try to extract from Kanamori's account an intrinsic reason for believing Replacement, not just extrinsic reasons, then the structuralist-style argument discussed above should really be characterized as not about sets, but classes. To speak of sets as the objects being structurally compared is loose. For sets in set theory are not comparable to, say, groups in group theory; universes are. If we are being precise, the justification should take the form: some classes are "large" and some classes are "small," and Replacement holds because if one class is small and another class is structurally similar (i.e., similarly sized), then the other class should also be small.

The difficulty is that it is not obvious that "smallness" is a structural property of classes. Classes are usually taken as intensional entities—if they are taken as entities at all. It is not completely clear that one can apply structuralist-style thinking to classes, in particular to draw coherent conclusions about their "size." Furthermore, if one is inclined to be structuralist at all, it is questionable whether set theory is an appropriate foundation to be using in the first place. There are other systems, such as category theory, that fit more harmoniously with structuralist intuitions (see, e.g., [Awo04] and [Lan99]).

As a further issue, Linnebo has argued that structuralism, at least in its stronger forms, is actually inconsistent with the iterative conception of set. The main incompatibility comes from the structuralist theses that individual objects in a structure are dependent on one another and/or dependent on the structure as a whole. If sets are formed according to the iterative conception, he says, then they should be dependent only on their members (and perhaps members of those members, etc.)—not on higher-ranked sets. It is especially odd to say that the singleton set $\{a\}$ depends on anything other than just a, especially anything high up in the hierarchy:

> This asymmetric dependence [of sets upon their members, and not vice versa] is in fact a very good thing, as there are all kinds of difficult questions about the higher reaches of the hierarchy of sets. How far does the hierarchy extend? Are the different stages rich enough for the continuum hypothesis to fail? It would be a pity if very simple sets, such as the empty set and its singleton, depended on the entire hierarchy of sets, and their identities could therefore not be completely known before these hard questions had been answered. But fortunately the situation is the reverse. In particular, we can give an exhaustive account of the identity of the empty set and its singleton without even mentioning infinite sets. [Lin08, p. 73]

Thus, to sum up, Kanamori's attitude towards axiomatic set theory is one that preferences extrinsic justification principles over intrinsic ones. He is not attached to the iterative conception as a sole foundational idea generating the set-theoretic universe. If Replacement is not amenable to intrinsic justification, then those who endorse the axiom (i.e., nearly everyone) must search elsewhere.

Of course, outside structuralism and limitation of size, there have been some recent attempts to revitalize arguments for Replacement based on reflection principles, which are arguably better motivated by the iterative conception. Such principles follow the intuition that the universe of sets is "a very rich structure with no conceivable end" [Dra74, p. 123], too large to be uniquely characterized, hence any property that might be ascribed to V must also be ascribable to some initial segment V_α. Some formalizations of this notion entail Replacement.[13]

But there are difficulties with this approach. Why are reflection principles justified? How do they follow from our concept of set? On Burgess' account, reflection is motivated by the principle, "The xs form a set unless ... they are indefinably or indescribably many" [Bur04, p. 205]. Put differently, the universe of sets is so large and comprehensive that it must be indescribable. But what is meant here by 'indescribable'? The answer given is that "any statement about all of [the members of the universe] continues to hold if reinterpreted to be not about all of them but just about some of them, few enough to form a set." In other words, there is no way to distinguish the universe from its initial segments by means of a first-order sentence that is true of the universe and not its earlier stages. In Drake's words, "it always seems a plausible step, in view of the reflection principle, to take a property of the whole universe ... and postulate that it already holds at some level V_α" [Dra74, p. 124]. But this answer is just a restatement of the principle of reflection, not a justification.

Another answer is that 'indescribable' means something else. But in that case, it's not clear *what* it means without further explanation. A robust account of indescribability is needed before reflection can properly ground axioms like Replacement. If the notion of indescribability is taken too generally, such as, "not describable or distinguishable by any broadly linguistic means," then the statement of reflection is clearly false: after all, the universe V contains all sets, but its initial segments V_α do not. So the challenge here is to formulate a notion of 'indescribable' that renders the justification of reflection neither question-begging nor false.

An additional hurdle is that some formulations of higher-order reflection are inconsistent. For instance, Koellner offers examples of reflection principles that are proved inconsistent in [Koe09]. Higher-order reflection principles, as noted by Reinhardt, are also inconsistent: "Since it is the presence of the classes as parameters in Bernays' schema which appears to be responsible for its great strength, we would eventually like to continue by formulating a third-order reflection principle which allows third-order objects as parameters. It does not appear to be possible to generalize ... in this direction very directly" [Rei74, p. 196]. While not every statement of reflection is inconsistent, one must meet the challenge of explaining why its justification, whatever that is, provides for the consistent versions and not the inconsistent ones.

Regardless of the status of arguments for reflection, I suggest that the principle of transfinite recursion is so natural and basic that it should not be held hostage to the success of these more ambitious programs. The approach I propose and pursue in the next sections is a new and yet entirely natural approach. Rather than begin with a strong principle such as reflection and use it to justify Replacement, here we begin with notions related to recursion and combine them with other principles to obtain Replacement, or at least fragments of it. So, on this picture, Replacement is not independently justified as a special instance of one larger principle, but rather as a consequence of several principles which we already believe.

[13] See, e.g., [Ber76] (of which a recent reconstruction is given in [Bur04]), building on [Lé60]. For a history of these, see [Kan06, Sec. 2] and [Kan09].

Even if this specific proposal does not work out, it seems like a natural and yet hitherto unexplored route: instead of justifying Replacement from 'above' by stronger principles, we try to build up to it from 'below' by a collection of principles with *primae facie* intrinsic support. Even if the specific proposal below encounters problems, it is surprising that this natural idea has not been pursued in the extant literature on the justification of principles like Replacement. Obviously, one of the problems which this proposal encounters is that it must argue that some recursion-like notions are built into our concept of set. This point is addressed further in sections 4-5. The next section is concerned with making precise the idea of building up from 'below' to versions of Replacement.

3. Calibrating Thresholds: Some Reverse Set Theory

Above we have motivated the question of how to justify transfinite recursion and explained why extant proposals are wanting. In this section, we'd like to come to some better understanding of how to formalize transfinite recursion as a single principle and investigate how it relates to other standard principles like Replacement.

Recall that Z is Zermelo's Z—the subsystem of ZFC obtained by removing the Axiom Schema of Replacement and the Axiom of Choice—and that ZC is just Z plus Choice[14]. Further, recall that these theories have obvious second-order variants, wherein the schemes are replaced by single axioms and wherein one adds the full second-order comprehension schema.[15]

Definition 1. Let *Ordinal Recursion* (Ω-Rec) be the following axiom in the signature of second-order ZC: for every function $G : V \to V$ there is a function $F : \Omega \to V$ such that for every ordinal α, the class $F \restriction \alpha$ forms a set and $F(\alpha) = G(F \restriction \alpha)$.

This is essentially the theorem of transfinite recursion stated as an axiom. As an attempt to define an axiomatic system that includes recursion without Replacement, Ω-Rec would seem a reasonable place to start. As Proposition 3 will show, though, we can recover Replacement from Ω-Rec, provided we also have the following principle.

Definition 2. Let *Ordinal Abstraction* (Ω-Abs) be the following axiom in the signature of second-order ZC: if X is a set and $(X, <)$ is a well-ordering, then there is an ordinal α and an order-preserving bijection $f : \alpha \to X$.

The axiom Ω-Abs is a way of formally expressing the natural idea that every well-ordered set determines an ordinal. Before Zermelo and Von Neumann, Cantor's conception of ordinal

[14]We actually will not be appealing to the Axiom of Choice as such, but rather the well-ordering theorem. For convenience, we will abbreviate this by C.

[15]While first-order treatments of set theory are more common, our aim is to formalize principles yielding recursion, which is then used to justify part of Replacement. Recursion is most naturally formulated as a second-order axiom. In future work, I hope to investigate whether these results can be replicated in first-order logic or related systems like Lavine's schematic logic [Lav99]. This would be important because there are at least two well-known concerns with the invocation of second-order logic in the context of set theory. First, some suggest that the standard semantics for second-order logic is already set-theoretically "entangled". (The metaphor of mathematical entanglement is originally from Charles Parsons. For further use of it, see [Koe10]). Second, there is the concern that an appeal to second-order logic is inconsistent with the iterative conception (see [Bur85]).

consisted of just an arbitrary representative of the class of well-orderings of a given order-type.[16] The name *Ordinal Abstraction* reflects the idea that we can abstract away from the particularities of the class of ordinals Ω as ordered by the $\in$-relation to any isomorphic well-order. If we believe this, then by Proposition 3 we simply recover ZFC:[17]

Proposition 3. *Against the background of second-order ZC, the following are equivalent:*

(1) *The Axiom of Replacement*
(2) *The conjunction of* Ω*-Rec and* Ω*-Abs*

Proof. First suppose the Axiom of Replacement, so that we are working in second-order ZFC. The usual proof of transfinite recursion from ZFC carries verbatim to show the axiom Ω-Rec holds, and similarly for the axiom Ω-Abs. Second, suppose that we are working in second-order ZC plus the axioms Ω-Rec and Ω-Abs. Let $H : X \to V$ be a class function with domain a set X. It must be shown that the range of H is also a set. Let $(X, <)$ be a well-ordering. By the axiom Ω-Abs, choose an order-preserving bijection $g : \gamma \to X$ for some ordinal γ.

Now define $G : V \to V$ as follows. If f is the graph of a (set-) function with domain $\alpha < \gamma$, then set $G(f)$ equal to the union of f with the singleton of $H(g(\alpha))$; otherwise, set G equal to zero. Then apply the axiom Ω-Rec to obtain the $F : \Omega \to V$ associated to G. Then $F \restriction \gamma$ forms a set, so that $F``\gamma$ forms a set, and we claim that $\cup(F``\gamma) = H``X$. Clearly by construction of $G$ and the relation between $F$ and $G$, we have $\cup(F``\gamma) \subseteq H``X$. Conversely, if $y = H(x)$ where $x \in X$, then since $g : \gamma \to X$ is a bijection, choose $\delta < \gamma$ with $g(\delta) = x$. Then we have that $y = H(x) = H(g(\delta)) \in F(\gamma)$ and hence that $y \in \cup(F``\gamma)$. $\square$

As an alternative to Ω-Rec, we now define σ-Rec, an axiom intended as a weakening of Ω-Rec that only affirms the existence of functions built by adding to the function's graph the suprema of previous values. This is a generalization of primitive recursion on the natural numbers.

Definition 4. Let *Supremum Recursion* (σ-Rec) be the following axiom in the signature of second-order ZC: for all ordinals γ_0 and all functions $G_s : \Omega \to \Omega$ there is a unique function $F : \Omega \to \Omega$ such that

$$F(0) = \gamma_0, \qquad F(\alpha + 1) = G_s(F(\alpha)), \qquad F(\lambda) = \sup_{\alpha<\lambda} F(\alpha) \tag{1}$$

wherein λ is a limit.

Remark 5. Note that ZC does not prove σ-Rec. For instance, consider the model $(V_{\omega+\omega}, \in)$, which satisfies ZC. Suppose it were a model of σ-Rec. Then define $\gamma_0 = \omega$ and $G_s(\alpha) = \alpha+1$. Let $F : \Omega \to \Omega$ be the function associated to γ_0 and G_s. Then $F(\omega) = \sup_{n<\omega}(\omega+n) = \omega+\omega$, which is not an element of $V_{\omega+\omega}$. This shows that σ-Rec is not already part of ZC.

Moreover, the system ZC + Ω-Abs does not prove σ-Rec. Let κ be the least fixed point of the $\beth$-function—that is, $\kappa = \beth_\kappa$. Then we can show that $(V_\kappa, \in) \models \text{ZC} + \Omega\text{-Abs} + \neg\sigma\text{-Rec}$. To show $(V_\kappa, \in) \models \Omega$-Abs, let $x \in V_\alpha$ for some $\alpha < \kappa$. Then in V there is an ordinal β and bijection $f : x \to \beta$. So $|\beta| = |x| \leq |V_\alpha| \leq \beth_\alpha$. Since $\alpha < \kappa$, we have $\beth_\alpha < \beth_\kappa = \kappa$,

[16]See, e.g., [Fin98].

[17]Though we do not do so in this paper, one might also be interested in defining a variation of Ω-Rec that works with general well-orderings instead of just ordinals, in such a way that the axiom Ω-Abs is not needed.

so $|\beta| < \kappa$. Thus $\beta < \kappa$, so $\beta \in V_\kappa$. Since x and β are both in V_κ, so is f by closure under Union, Pairing, and Power Set.

Now suppose for contradiction that $(V_\kappa, \in) \models \sigma$-Rec. Then there exists a function F given by:

$$F(0) = \omega \qquad F(\alpha+1) = \beth_{F(\alpha)} \qquad F(\lambda) = \sup_{\alpha<\lambda} F(\gamma).$$

Now we claim that $F(\omega) = \beth_{F(\omega)}$, contradicting the supposition that κ is minimal. For $F(\omega) = \sup_{n<\omega} F(n)$, and hence $\beth_{F(\omega)} = \beth_{\sup_{n<\omega} F(n)} = \sup_{n<\omega} \beth_{F(n)} = \sup_{n<\omega} F(n+1) = \sup_{n<\omega} F(n) = F(\omega)$. This completes the argument.

One can show further that while κ is known to be a singular cardinal (since it is the limit of the ω-sequence $\{\beth_0, \beth_{\beth_0}, \beth_{\beth_{\beth_0}}, \ldots\}$), any regular cardinal κ' for which $(V_{\kappa'}, \in) \models \mathrm{ZC} + \Omega$-Abs is inaccessible. One can show this by letting $\lambda < \kappa'$. Then we have $\lambda \in V_{\kappa'}$, so $\mathcal{P}(\lambda) \in V_{\kappa'}$. By C, there is a well-order $(\mathcal{P}(\lambda'), <)$, so by Ω-Abs there is an $\alpha \in V_{\kappa'}$ that is order-isomorphic to the well-order. So $|\mathcal{P}(\lambda)| = |\alpha| < \kappa'$.

Remark 6. Unlike Ω-Rec, the axiom σ-Rec does not imply Replacement in ZC + Ω-Abs (assuming consistency), which we show below in Proposition 12. It is, however, equivalent to what could be called Ω-Replacement:*

Definition 7. Let Ω-Replacement, or Ω-Rep, be the axiom asserting that for every α, the range of every function $G' \colon \alpha \to \Omega$ is a set.

Proposition 8. *In ZC + Ω-Abs, the axiom σ-Rec is equivalent to Ω-Replacement.*

Proof. For the right-to-left direction, a similar argument as in the usual proof of transfinite recursion shows the existence of functions defined by σ-Rec using just the fragment of Replacement found in Ω-Rep. For the other direction, suppose that the sentences of $\mathrm{ZC} + \Omega$-Abs $+ \sigma$-Rec are true. Let α be an ordinal and $G' \colon \alpha \to \Omega$. What we must show is that the range of (G') exists as a set. Define $G \colon \Omega \to \Omega$ by letting $G(\delta)$ be the least ordinal above δ in the range of G', or 0 if there are no such. Suppose for contradiction that the proposition were false and $\mathrm{rng}(G')$ were a proper class. By the ZC axioms, the universe is closed under Separation, Union, Pairing, and Power Set, so the function G is increasing unboundedly in Ω (rather than eventually reaching a point γ at which $G(\delta) = 0$ for all $\delta \geq \gamma$). But this would then imply that the following function definable by σ-Rec is strictly increasing unboundedly in Ω:

$$F(0) = \min(\mathrm{rng}(G')), \quad F(\delta+1) = G(F(\delta)), \quad F(\lambda) = \sup_{\delta<\lambda} F(\delta).$$

By Power Set, there exists a cardinality $|\alpha|^+$ immediately above $|\alpha|$. The axiom σ-Rec ensures that $F \restriction |\alpha|^+$ is a set function, and hence so is $F \restriction |\alpha|^+ \setminus \{\text{limits}\}$, the restriction of F to 0 and successor ordinals of size bounded by $|\alpha|^+$. Note that $F \restriction |\alpha|^+ \setminus \{\text{limits}\}$ injects into $\mathrm{rng}(G')$. (The same cannot be said for $F \restriction |\alpha|^+$, since the images of the limit ordinals are not in general in $\mathrm{rng}(G')$.) From this injection one can now obtain a contradiction by constructing an injection from $|\alpha|^+ \setminus \{\text{limits}\}$ to α. Let $H \colon \mathrm{rng}(G') \to \alpha$ be given by $H(\zeta) = \min(\{\theta \mid G'(\theta) = \zeta\})$. Then $H \circ F \restriction |\alpha|^+ \setminus \{\text{limits}\}$ injects into α. $\square$

Definition 9. Let the *Axiom of Ordinals* denote the assertion that V_α exists for every α.

*Thanks are owed to Sam Roberts. See Section 6.

This name is found in Potter's [Pot04, p. 218]. There, as in our setting, the Axiom of Ordinals is defined in a system without the Axiom of Replacement.

Proposition 10. *In ZC + Ω-Abs, the Axiom of Ordinals + σ-Rec + "$\forall x \exists \alpha\; x \in V_\alpha$" together imply Replacement.*

Proof. Suppose these axioms hold. Then let $H : X \to V$ be a class function with domain a set X. We must show that the range of H is also a set. By Ω-Abs, let $g : \gamma \to X$ be an order-preserving bijection for some ordinal γ. So $H \circ g \colon \gamma \to V$. Let $R\colon \mathrm{rng}(H) \to \Omega$ be defined by $R(s) = \mathrm{rnk}(s)$ (the rank of s exists by assumption). So $R \circ H \circ g\colon \gamma \to \Omega$. By σ-Rec and Proposition 8, we have Ω-Rep, so $\mathrm{rng}(R \circ H \circ g)$ exists as a set. Hence its supremum β exists as a set, which means $\beta + \omega$ exists. Thus $V_{\beta+\omega} =_{\mathrm{df}} \bigcup_{\lambda<\beta+\omega} V_\lambda$ exists by the axioms and contains every member of $\mathrm{rng}(H)$. □

We have seen now that σ-Rec entails the existence of many ordinals that otherwise would not necessarily exist, precisely the ones that exist under a weakened version of Replacement restricted to ordinal functions. Since the converse of Proposition 10 is obviously true, these observations suggest a way to characterize what the gap is between σ-Rec/Ω-Replacement and full Ω-Rec/Replacement, from the perspective of ZC + Ω-Abs—namely, the "width" of the universe, as measured by the existence of all V_α (for every ordinal α), which together contain all sets. But given a sufficiently "slim" model, in the spirit of [Mat01], we can show that ZC + σ-Rec + Ω-Abs alone cannot imply Replacement:

Definition 11. For finite α, let $D_\alpha = V_\alpha$. For $\alpha \geq \omega$, let $D_{\alpha+1} = \{x \mid \exists y, y' \in D_\alpha$ s.t. $x = \mathcal{P}(y) \;\vee\; x \subseteq y \;\vee\; x = \bigcup y \;\vee\; x = \{y, y'\}\}$ and let $D_\lambda = \lambda \cup \bigcup_{\alpha<\lambda} D\gamma$.

Proposition 12. *For any inaccessible $\kappa > \omega$, $D_\kappa \models$ ZC + σ-Rec + Ω-Abs + $\neg$Replacement.*

Proof. Since D_κ is closed under Union, Pairing, and Power Set, it is straightforward to see that $D_\kappa \models$ ZC. Further, since κ is inaccessible and all the ordinals of V_κ are in D_κ, we have $D_\kappa \models \Omega$-Abs. Likewise, the regularity of κ guarantees Ω-Replacement and hence also σ-Rec (indeed, σ-Rec follows from closure under suprema). It remains to be shown that D_κ fails Replacement. We argue that for every $x \in D_\kappa$, there exist only finitely many Zermelo ordinals[18] in its transitive closure $tc(x)$; thus $V_\omega \notin D_\kappa$ and Replacement fails. Suppose this were false. Then let α be the least ordinal such that D_α contains a counterexample x. Note that α cannot be a limit, for at limits the only new element added is a Von Neumann ordinal, which contains exactly two Zermelo ordinals ($\emptyset$ and $\{\emptyset\}$, neither of which is new anyway) and is identical to its own transitive closure. So $\alpha = \beta + 1$ for some β. Thus $x = \mathcal{P}(y)$ or $x \subseteq y$ or $x = \bigcup y$ or $x = \{y, y'\}$ for some $y, y' \in D_\beta$. But since $tc(x)$ contains infinitely many Zermelo ordinals, so must y or y', contradicting the minimality of α. □

To summarize, in this section we have considered two recursion principles as axioms: Ordinal Recursion (Ω-Rec), which is just the theorem of transfinite recursion stated as an axiom, and Supremum Recursion (σ-Rec), a weakened version. We showed that the first of these is nearly strong enough by itself to prove Replacement in ZC; all one needs to add is Ordinal Abstraction, the principle that there is an ordinal corresponding to every well-ordering. Then we showed that σ-Rec is equivalent to a fragment of Replacement, namely

[18]For reference, the Von Neumann ordinal $\alpha + 1 =_{\mathrm{df}} \alpha \cup \{\alpha\}$, whereas the Zermelo ordinal $\alpha + 1 =_{\mathrm{df}} \{\alpha\}$. So, e.g., the set of finite Zermelo ordinals is $\{\{\}, \{\{\}\}, \{\{\{\}\}\}, \ldots\}$.

Replacement on ordinals. We saw that full Replacement can then be recovered from this weaker recursion principle by further adding "V_α exists for every α" (the Axiom of Ordinals) and the basic assumption that every set appears on some stage. These results collectively provide reason to believe that recursion principles are plausible candidates for foundational axioms of a workable set theory. In the section that follows, we consider a third notion of recursion. Even if it should turn out that these particular formalizations fall short in some way, it is suggested here that some alternate approach within the same spirit should be promising.

4. Transfinite Computation and Algorithms Determine Recursions

Having seen the strength and the weakness of Supremum Recursion (Definition 4), we now look at a related and potentially useful notion from Jensen and Karp [JK71]:

Definition 13 (Jensen, Karp 1971)**.** A function $f : \Omega \to \Omega$ is a *primitive recursive ordinal function* iff it is generated by the following scheme:

- Initial Functions: $P_{n,i}(x_1, ..., x_n) = x_i$ for $1 \leq i \leq n < \omega), F(x) = 0, F(x) = x + 1$
- Case Distinction: $C(x, y, u, v) = x$ if $u < v$, else $C(x, y, u, v) = y$
- Substitution: $F(x, y) = G(x, H(y), y), F(x, y) = G(H(y), y)$
- Primitive Recursion $F(z, x) = G(\bigcup\{F(u, x) \mid u \in z\}, z, x)$.

Further, f is *ordinal recursive* iff it is generated by the above plus:

- Minimization Rule: $F(x) = min\{\xi \mid G(\xi, x) = 0\}$, provided $\forall x \exists \xi \; G(\xi, x) = 0$.

These are clearly intended as generalizations of primitive and total recursion on the natural numbers, respectively. Following this definition, a result is obtained that succinctly characterizes the ordinal recursive functions:

Theorem 14 (Jensen, Karp 1971)**.** *$f : \Omega \to \Omega$ is ordinal recursive iff it is $\Delta_1(L)$—i.e., it is definable by a Δ_1 formula with quantifiers ranging over L.*

A set-theoretic formula is Δ_1 if it is equivalent to some Σ_1 formula (a formula with only one unbounded existential quantifier) and also to some Π_1 formula (a formula with only one unbounded universal quantifier).

Theorem 14 now gives us two ways of generalizing upon the finitary notion of primitive recursion into a transfinite setting: σ-Rec (Definition 4) and the Jensen-Karp primitive recursive ordinal functions. We also have a definition of *ordinal recursion* that generalizes not just on the finitary notion of primitive recursion but on the full recursive functions of the sort computable by a Turing machine (primitive recursion being of course just a subset of these).

In a recent, rapidly growing body of literature, there have been a number of models of computation generalizing upon Turing machines into a transfinite setting, among which is the *ordinal Turing machine* (OTM) model of P. Koepke. It turns out that by a recent result of Koepke and Seyfferth ([KS09, Thm. 7]), the notion of ordinal recursive functions coincides precisely with the functions computable by an OTM. That is, a function is ordinal recursive if and only if it is computable by an OTM, given finitely many ordinal parameters. Given our guiding question, "What principles of recursion are available for the transfinite?", such machines are worth considering further.

In broadest terms, the workings of an OTM are relatively simple to picture. Rather than computing functions on the natural numbers, as a Turing machine does, it computes on ordinals. The below diagrams from [Koe05] demonstrate the difference. Figure 1 depicts a Turing machine computation, with the vertical axis representing time and each row representing the tape content after n number of steps. The bolded number indicates where the head is positioned. In Figure 2, a similar representation of an OTM shows the computation continuing through θ many steps and eventually stopping after some further ordinal number (unless the computation is non-halting). A formal definition of how the machine operates is available in [Koe05].

			SPACE								
		0	1	2	3	4	5	6	7	…	…
	0	**1**	0	0	1	1	1	0	0	0	0
TIME	1	0	**0**	0	1	1	1	0	0		
	2	0	0	**0**	1	1	1	0	0		
	3	0	**0**	1	1	1	1	0	0		
	4	0	1	**1**	1	1	1	0	0		
	⋮										
	n	1	1	1	1	**0**	1	1	1		
	$n+1$	1	1	1	1	1	**1**	1	1		
	⋮										

Figure 1. A Turing Machine Computation

		Ordinal Space …														
		0	1	2	3	4	5	6	7	…	…	ω	…	α	…
Ordinal Time …	0	**1**	1	0	1	0	0	1	1	…	…	1	…	1	0
	1	0	**1**	0	1	0	0	1	1			1			
	2	0	0	**0**	1	0	0	1	1			1			
	3	0	**0**	0	1	0	0	1	1			1			
	4	0	0	**0**	0	0	0	1	1			1			
	⋮														
	n	1	1	1	1	0	1	0	**1**			1			
	n+1	1	1	1	1	1	1	**0**	1			1			
	⋮	⋮	⋮	⋮	⋮	⋮									
	ω	0	0	1	0	0	0	1	1	…	…	**1**			
	$\omega+1$	**0**	0	1	0	0	0	1	1			0			
	⋮														
	θ	1	0	0	1	1	1	1	0	…	…	…	…	**0**	…
	⋮			⋮			⋮			⋮	⋮				
	⋮														

Figure 2. An Ordinal Turing Machine Computation

Given the equivalence between ordinal Turing machines and ordinal recursive functions, and in light of our earlier characterization of recursion as a type of "iterated procedure," a new answer to the question of justifying recursion suggests itself: perhaps set-theoretic functions ought to be understood algorithmically. The idea is to identify an intrinsically justified axiom or axiom schema linking the iterative conception of set with algorithms, either in the OTM sense or otherwise. It is natural to picture the process of generating new ranks of the cumulative hierarchy as an essentially algorithmic procedure.

The idea of identifying algorithms as the source of corresponding functions is not new. In 1971, Kreisel suggested the possibility of privileging calculations over the functions so determined by these calculations:

> Which comes first: the calculations or the functions that result from the calculations? [In the "calculations first" approach], the processes or rules come first, ideally determined by a phenomenological analysis of possible instructions (understandable by the means considered); the functions are defined in terms of these processes, and one may look for other systems of rules which compute the functions of the same class. Evidently one must expect a possible conflict of interest here: for a complete *description* of possible processes we want lots of rules; for *deciding* some specific question about the computable functions we often want few rules (provided only that they generate all the functions) ([Kre71] p. 151).

Two natural questions present themselves: first, is the claim that algorithms determine functions even formalizable in an axiomatic language? Second, if it is, how do we avoid the "conflict of interest" that Kreisel mentions, namely, the issue of the "completeness" of the axioms?

What Kreisel means by the possible conflict seems to be: in axiomatizations we face the conflicting choice between trying to have a manageable and understandable axiom-set on the one hand, and a complete axiom-set on the other hand. The axioms of Peano Arithmetic are effective and very easy to understand, but they are incomplete. We could, if we wanted, change the system so that the set of axioms is just simply *the set of all true sentences about numbers in the language of number theory.* But this would be a useless axiom system. For starters, the axiom set is not effectively computable; we have no way of knowing from the description whether a given sentence is an axiom or not. In fact, if we did—that is, if we were smart enough to know antecedently of every number-theoretic sentence whether it is true—then we would have no need for an axiom system. Hence the desire for completeness on the one hand, and for usefulness on the other, stand in tension. How should we balance these?

In the case of number theory, the prevailing formal system errs on the side of effectiveness rather than completeness. The same is true for set theory. It is notable that the idea of using the set of second-order validities as axioms for set theory to settle problems like CH has not been seriously pursued—nor has some comparable but weaker system been seriously proposed—because there is no decision procedure to determine which formulas belong within that set. But in the case of computation, we do not have a canonical axiom system to consider.

It turns out that Kreisel's own answer to the original question, "Which comes first... ?" is actually that functions come before calculations. Taking the notion of a recursive function in the finitary setting and looking at transfinite analogues, Kreisel's brand of generalized recursion works its way up the constructible hierarchy and constructs higher recursive functions through definability criteria (see, e.g., [Kre61] and [KS65]). The literature in this field, such as presented in Sacks' [Sac90], clearly prioritizes functions. But Kreisel is fully aware of the possibility of what he calls *metacomputation*, taking a machine-theoretic approach that prioritizes calculations. He suggests that this possibility is "controversial":

> Here it is to be remarked that, in connection with metarecursion theory, I intentionally excluded any discussion of the notion of 'metacomputation' for conceptual analysis ... There was no need to bring in controversial matters. [Kre71, p. 180]

In light of OTMs and other models of higher computation, surely the idea is far less controversial now. At the time, Kreisel's main challenge was:

> Is there such a thing as a general concept of computation? [O]r the related question: Is there such a thing as an extension of Church's Thesis to general (abstract) structures? ... If the answer is positive, we have to do with the *analysis* of a concept, specifically a phenomenological analysis of the kind given by Turing for mechanical computations of number theoretic functions. [Kre71, p. 144]

To place higher algorithms in a position prior to the functions they define, he says, we must have a conceptual analysis akin to Church's Thesis, with a phenomenological description of *what it is like* to be a computer.[19]

It just so happens that there is now a project being developed in precisely this direction. Though not evidently with Kreisel in mind, Merlin Carl formulates and defends in [Car13] a version of Church's Thesis that is extended to include transfinite models of computation, specifically the OTM model. He articulates a pre-theoretic concept of higher computation, conducted by what he calls an Idealized Agent Machine (IAM). First he presents a philosophical argument that the IAM model is computationally equivalent to Kitcher's notion of an idealized mathematician. Then he proves formally that an IAM is equivalent to an OTM. The phenomenology of an IAM is described thoroughly:

> At each time, he has a complete memory of his earlier computational activity. Also, he has a working memory where he may store information. We assume that the working memory consists of separate places, each containing one symbol from a finite alphabet. The agent is working [according to] instructions that determine his activity ... given by some finite expressions. [Car13, p. 6]

These descriptions are later on made more precise. Included in the IAM definition is also the ability to look back at the course of past computation and ask questions of it, especially logically complex ones such as, "Does there exist a time such that, for all times after, the value of cell α did not change?" The answers to such questions are critical for taking inferior limits of cell values, which is needed for emulating the operations of an OTM.[20] In the next

[19] Turing's original definition of a 'computer' in [Tur36] depicted a human being, not a machine, following the rote instructions of a program in "a desultory manner". For this reason, Sieg [Sie05] writes 'computor' to indicate a human agent calculating in this way and 'computer' to indicate a machine in the word's contemporary sense. In these terms, Kreisel's challenge is to provide a general concept of computation and an associated Church's Thesis, while also offering an account of the phenomenology of such a comput*or*.

[20] An OTM works similarly to a classical Turing machine at successor ordinal-numbered steps of computation, but at limit ordinal steps the cells' contents must be calculated by taking some sort of limit of previous values,

section, Carl discusses the agent's perception and use of time and space, and argues that the correct way to understand these in both cases is by considering them as discretized, ordered units, indexed by the transfinite ordinals. From this paper, and the rest of the literature on transfinite computation, we have an answer to Kreisel's once open question:

> Mechanical [i.e., finitary] operations are appropriate (fitting) if the objects on which we operate are given in a finitistic (spatio-temporal) way. ... The question ... concerning the existence of ... a general concept of computation becomes the question whether, on examination, it will be possible to isolate a general notion from the mixture which we associate with the idea of operations proper to (or: implicit in) the nature of the objects, and whether this (hypothetical) notion admits of a useful theory ([Kre71] p. 179-80).

Kreisel claims here that just as we have a solid criterion for evaluating whether we can compute on some given objects in Turing's sense—namely, the objects must be finitary—so too do we need to have an equally solid criterion for whether a given object is appropriate for computing upon in a generalized (transfinite) sense. It is evident that we do now have such a criterion. Corresponding to *finitary/finitistic* in the setting of Turing machines, we now have *set-sized*, or perhaps *ordinal-sized*, in the setting of OTMs/IAMs. And this leads to a natural idea of ordinal computation. (Of course, with Choice there is no great distinction between saying something is set sized or ordinal sized, but the latter puts emphasis on the fact that the objects are ordered, which coheres with Carl's argument that infinitary space and time are best represented by ordinals.)

All this is to say that we are in a much better position now to entertain the notion that algorithms determine functions than we were a few decades ago.[21] Before exploring this possibility further, however, I want to address a potential objection. One might suggest that the claim, "Algorithms determine functions," is suspect because of the "determination" notion contained therein. What does it really mean to say that an algorithm *determines* a function? Consider, though, the analogous principle of Ordinal Abstraction (Ω-Abs) from Definition 2 of section 3. This axiom naturally captures the intuitive thought that any well-order determines an ordinal. If we wanted, we could formalize this thought in a way that entirely avoids any "determination" language. In particular, we can recast the determination talk in terms of a function from one thing to another that preserves the relevant properties. This possibility suggests that blanket scepticism about determination talk is misplaced.

So here is a first attempt at formalizing the claim in the particular form, "OTM algorithms determine ordinal functions." One might simply try to describe, in patently set-theoretic language, the workings of the ordinal Turing machines, using the usual set-theoretic reductions, and use this to write down axioms that effectively say, "If such-and-such OTM algorithm working on the ordinals is total, then there is a corresponding function on ordinals that has the same values." The difficulty with this proposal is that in defining "halting computation," we must use transfinite recursion, or at least some significant fragment of it. Defining how the OTM computes cell values after a limit ordinal number of operations requires taking the inferior limits of all prior cell values, and the same is true of the machine's head position

implemented here as the *lim inf*. This means the machine must be able to answer for each cell the Σ_2 question of whether there *exists* a step before the limit such that, *for all* steps after, the cell value was stabilized at 1. If so, the cell's limit value becomes 1, and otherwise (i.e., if it diverged or it stabilized at 0) it is set to 0.

[21]Since not every computation halts, it is more precise really to say that *some* algorithms determine functions.

and program state. So if we are working in ZC and looking for a principle to add, we cannot directly express the notion of halting computation without already having access to more recursion than is available in the theory. Not only does ZC not include recursion, it also has models without ordinals $\omega + \omega$ or larger; hence there is no clear reason to believe it makes sense to consider a machine with tape or computational run-time that long. In short, ZC is too weak to talk about OTMs.

An alternative approach would be to try to avoid this issue by referring not to machine computations directly, but rather to the complexity of the sets computable by such machines. As mentioned earlier, we have that OTM computability coincides precisely with ordinal recursivity. This comes from Theorem 14 together with the following result:

Theorem 15 (Koepke, Seyfferth 2009)**.** *A set of ordinals is OTM-computable if and only if it is $\Delta_1(L)$—i.e., it is definable by a Δ_1 formula with quantifiers ranging over L.*

This theorem is the natural analogue in the OTM setting of the folklore result that a function on natural numbers is Turing-computable if and only if its graph is Δ_1-definable over the standard model of the natural numbers.[22] Using this result (and the same definition of $\Delta_1(L)$ as after Theorem 14), the proposal is that for each Σ_1-formula $\varphi(x) \equiv \exists\, y\; \varphi_0(x, y)$ and Π_1-formula $\psi(x) \equiv \forall\, y\; \psi_0(x, y)$, we state an axiom of the following form:

$$\begin{aligned} &[[\forall x\ (L, \in) \models \varphi(x) \leftrightarrow \psi(x)] \quad \& \quad [\forall \alpha\ \exists!\, \beta\ (L, \in) \models \varphi(\langle \alpha, \beta \rangle)]] \\ &\qquad\qquad\qquad\qquad \Longrightarrow \quad [\exists F\ \forall \alpha, \beta\ F(\alpha) = \beta \leftrightarrow \varphi(\langle \alpha, \beta \rangle)] \end{aligned} \tag{2}$$

Unfortunately for this proposal, the antecedent of the axiom contains reference to the inner model L and what holds true and false within it. If we are considering new axioms for ZC, which does not already prove transfinite recursion, then there is no obvious way to express the class L, as L is defined transfinite-recursively. So the same circularity problem appears as before. One attempted response is to suggest we simply replace L by V, for one can easily refer to all sets in V without appeal to recursion. The problem with this suggestion is that we don't know that the resulting axiom schema is actually true.

Another attempt would be to move the arrow in the conditional to outside the theory. That is, let us consider the theory consisting of all sentences $\exists F\ \forall \alpha, \beta\ F(\alpha) = \beta \leftrightarrow \varphi(\langle \alpha, \beta \rangle)$ such that the two conjuncts in (2) above hold. The difficulty is that while this is technically a theory—it is a set of sentences—it is (highly) non-computable, and so it runs us into the second horn of Kreisel's dilemma. Here, then, we have a theory that is not a system you and I can reasonably adopt.

The challenges with expressing formally the idea that algorithms determine functions on the approaches mentioned so far suggest that a different type of strategy is called for. This paper will not develop a formal system that definitively settles the question of how to do this. But one promising idea is to articulate a formal language in which the notion, "OTM-algorithm e halts on input α and returns β," is treated as primitive. The proposal, then, is to develop axioms for this primitive notion that deliver recursion principles such as σ-Rec, Ω-Rec, or similar. It turns out that such a procedure has already been developed in the computability theory literature for Turing computation in [Bee85, Ch. VI], using PCAs (partial combinatory algebras). In the setting of OTMs, we can emulate much of this same

[22]See, e.g., [Soa87, Ch. 2 and Ch. 4].

method. The merit of this approach is that it circumvents the formal worries above. The downside is that we must find a way to philosophically motivate the primitive.

5. Transfinite Algorithms and the Iterative Conception

We have looked at a few basic proposals for how to formalize the notion that algorithms determine functions. These particular attempts may or may not lead us in the right direction, but the project overall is important if our goal is to give an intrinsically justified principle behind transfinite recursion. The hope is that unlike Replacement, recursion can be justified directly from the iterative conception by appeal to the algorithmic nature of the formation of stages in the universe.

This idea has not been deeply explored, but its roots can be found in canonical texts on the iterative conception. In [Boo71], for instance, Boolos invokes what greatly resemble algorithmic notions in his description of the intuitive picture behind the iterative conception. Describing how the stages are formed, he repeatedly uses recursive language:

> At stage one, form all possible collections of individuals and sets formed at stage zero. [...] At stage two, form all possible collections of individuals, sets formed at stage zero, and sets formed at stage one. At stage three, form all possible collections of individuals and sets formed at stages zero, one, and two. At stage four, form all possible collections of individuals and sets formed at stages zero, one, two, and three. Keep on going in this way [...]. [Boo71, p. 221]

The phrase, "Keep on going in this way," appears twice more on the page: once after the discussion of stages $\omega+1, \omega+2, \omega+3, \ldots$, and once after stages $\omega\times 2, \omega\times 3, \ldots, \omega\times\omega, \ldots$. This manner of speaking comes out very naturally in this setting.[23] What is being described is an iterated procedure of set formation—in short, a recursion.

In the later [Boo89], he writes:

> The fact that it takes time to give such a sketch, and that certain sets will be mentioned before others might easily enough be (mis-) taken for a quasi-temporal feature of sets themselves, and one might be tempted to say that sets coming earlier in the description actually *come earlier*, that sets cannot exist *until* their members do, that they *come into being* only after their members do, and that they are *formed* after all their members are.
>
> In any case, for the purpose of explaining the conception, the metaphor is thoroughly unnecessary, for we can say instead: there are the null set and the set containing just the null set, sets of all those, sets of all *those*, sets of all *Those*, ... There are also sets of all **THOSE**. Let us now refer to these sets as "those." Then there are sets of those, sets of *those*, ... Notice that the dots "..." of ellipsis, like "etc.," are a demonstrative; both mean: *and so forth,* i.e., in *this* manner forth.

Boolos spells out the connection between sets and recursive procedures even more explicitly in a description of Gödel's views from 1951:

> He then proceeds to lay out the "iterative" or "cumulative" hierarchy of sets: we begin with the integers and iterate the power-set operation through the

[23]The phrase chosen later in [Boo89] is "And so it goes," but the meaning is much the same.

> finite ordinals. This iteration is an instance of a general *procedure* for obtaining sets from a set A and well-ordering R: starting with A, iterate the power-set operation through all ordinals less than the order type of R (taking unions at limit ordinals). Specializing R to a well-ordering of A (perhaps one whose ordinal is the cardinality of A) yields a *new operation* whose value at any set A is the set of all sets obtained from A at some stage of this procedure, a set far larger than the power set of A. We can require that this new operation, and indeed *any set-theoretic operation, can be so iterated,* and that there should also always exist a set closed under our *iterative procedure* when applied to any such operation. [Gö01, p. 291, emphasis added]

In this light, it should not be so foreign to think that recursive algorithms lie at the heart of our very conception of set. Indeed, I believe it is no accident that the 1971 formalization of the conception results in a system producing precisely the recursive ordinals. As recounted in [Boo89]: "In 'The Iterative Conception of Set', I claimed that not even the existence of a stage corresponding to the first non-recursive ordinal is guaranteed by a formalization of the iterative conception and therefore that replacement does not follow from the iterative conception."[24] There are many ordinals that could have been the stopping point of the iterative universe without implying Replacement: $\omega + \omega$, for one. Any ordinal below the least fixed point of the $\beth$-function, or even any accessible cardinal, would have done the trick. But it is telling that the upper bound was instead the first non-recursive ordinal. If recursive algorithmic notions have nothing to do with our conception of set, then this fact constitutes a very striking coincidence.

In the last section, we saw some initial attempts to express formally that (some) algorithms determine functions. The difficulty encountered in that setting, without access to Replacement, is reminiscent of the difficulty one has in trying to express the meta-principle that (some) classes determine sets. The latter is patently true, but also difficult to make formal; it is not expressible in first-order ZFC, for instance, because classes are not referable objects in the theory. As we have seen, the former is similarly difficult to make formal in ZC, because algorithms and equivalent notions seem to be dependent on recursion. Each of these principles is thus a pre-theoretic intuition whose formalization is non-trivial.

There is substantial parallel between the two. If we consider how an algorithm that fails to determine a total function fails, we can see it does so in one of two ways: either it fails to produce a total function because some inputs cause it to loop infinitely, or else because some inputs provoke a never-ending search for some unfulfillable halting condition (e.g., in the finite case, a search for an even prime greater than 2). When a class fails to determine a set, the failure can likewise be characterized in some cases as a kind of looping behavior, and in other cases as a neverending search. The Russell class $R \coloneqq \{x \mid \neg x \in x\}$, for instance, can on this picture be described as failing to determine a set not because it is too large, as in the limitation of size conception, but rather because an algorithmic procedure that attempts to produce it must loop once it asks whether $R \in R$. The answer depends on whether $\neg R \in R$, which in turn depends on the answer to whether $R \in R$. Any set of instructions we may intuitively imagine giving a transfinite computer that would potentially produce R

[24] Here Boolos is referring to [Boo71, fn. 13], which says that R_{δ_1} models the given sketch of the iterative conception, where δ_1 is the first non-recursive ordinal (commonly denoted ω_1^{CK}—see [Sac90, p. 10]).

as a set would only cause the device to loop. A class, on this construal, is a recipe for the construction of a set, and some recipes (algorithms) just don't produce an output.

Another canonical example of a class that doesn't determine a set is Ω, bringing to mind Burali-Forti. The failure to produce a set in this case can be diagnosed as an unending search for an ordinal that does not exist. Because Ω would have to contain Ω, as the well-worn paradox goes, it would have the property $\Omega < \Omega$, which is provably impossible. A transfinite algorithm that attempts to compute a code for such a set would require Ω-many steps to complete the task, and thus cannot do so. By analogy, a Turing machine cannot in finitely many steps produce a code for ω, and any attempt to do so would classically be characterized as "going on forever." The class Ω, by its very nature, takes longer to create than any halting algorithm can possibly run.

If we incorporate into the iterative conception an algorithmic concept of set construction, we are provided with a way to understand why classes can fail to produce sets. The fusion of algorithmic procedures into the iterative conception is, I suggest, a natural and well-motivated way to do this.

Depending on how this can be done, there are many possible recursion principles that could come out of such a theory. If a sufficiently strong principle of recursion is available, then by Proposition 3 in section 3, or a similar result, we may derive Replacement as a consequence. Conversely, if only some weaker principle like σ-Rec is available, then the theory will not contain all of Zermelo-Fraenkel set theory ZFC. Still, this would be a significant step forward from existing formalizations of the iterative conception with strength on the order of Zermelo's set theory Z.

6. Acknowledgments

I would like to acknowledge Kyle Banick, Philip Ehrlich, Ethan Galebach, Peter Koepke, Christopher Menzel, Charles Parsons, Erich Reck, Sam Roberts, Sam Sanders, Claudio Ternullo, Sean Walsh, Kai Wehmeier, and John Wigglesworth, for their numerous helpful suggestions. Propositions 8, 10, and 12 are due to Roberts. I also thank the referees for their many written comments. Finally, I wish to thank Benedikt Löwe, Dirk Schlimm, and the other organizers of FotFS VIII. The research for this paper was partly funded by a grant from the Analysis Trust.

References

[Awo04] Steve Awodey. An answer to Hellman's question: 'does category theory provide a framework for mathematical structuralism?'. Philosophia Mathematica, 12(1):54–64, 2004.

[Bee85] Michael J. Beeson. Foundations of Constructive Mathematics, volume 6 of Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, Berlin, 1985.

[Ber76] Paul Bernays. On the problem of schemata of infinity in axiomatic set theory. In Gert H. Müller, editor, Sets and Classes on The Work by Paul Bernays, volume 84 of Studies in Logic and the Foundations of Mathematics, pages 121 – 172. Elsevier, 1976.

[Boo71] George Boolos. The iterative conception of set. Journal of Philosophy, 68(8):215–231, 1971.

[Boo89] George Boolos. Iteration Again. Philosophical Topics, 17:5–21, 1989. Reprinted in [Boo98].

[Boo98] George Boolos. Logic, Logic, and Logic. Harvard University Press, Cambridge, MA, 1998. Edited by Richard Jeffrey.

[Bur85] John P. Burgess. Reviews. The Journal of Symbolic Logic, 50:544–547, 6 1985.

[Bur04] John P. Burgess. *E pluribus unum*: Plural Logic and Set Theory. Philosophia Mathematica, 12(3):193–221, 2004.

[Car13] M. Carl. Towards a Church-Turing-thesis for infinitary computations. Preprint, 2013.

[Dra74] Frank R. Drake. Set Theory: An Introduction to Large Cardinals, volume 76 of Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam, 1974.

[Fin98] Kit Fine. Cantorian Abstraction: A Reconstruction and Defense. The Journal of Philosophy, 95(12):599–634, 1998.

[GÖ1] Kurt Gödel. Kurt Gödel, Collected Works, Volume III: Unpublished Essays and Lectures. Oxford University Press, USA, May 2001.

[Hal84] Michael Hallett. Cantorian Set Theory and Limitation of Size, volume 10 of Oxford Logic Guides. The Clarendon Press, New York, 1984.

[Hor11] Leon Horsten. The Tarskian Turn. The MIT Press, Cambridge, 2011.

[Jec03] Thomas Jech. Set Theory. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The Third Millennium Edition.

[JK71] Ronald B. Jensen and Carol Karp. Primitive recursive set functions. In Dana S. Scott, editor, Axiomatic Set Theory, volume XIII (Part 1) of Proceedings of Symposia in Pure Mathematics, pages 143–176, Providence, Rhode Island, 1971. American Mathematical Society.

[Kan06] Akihiro Kanamori. Levy and set theory. Annals of Pure and Applied Logic, 140(1):233–252, 2006.

[Kan09] Akihiro Kanamori. Bernays and set theory. Bulletin of Symbolic Logic, 15(1):43–69, 03 2009.

[Kan12] Akihiro Kanamori. In praise of replacement. Bulletin of Symbolic Logic, 18(1):46–90, 03 2012.

[Koe05] Peter Koepke. Computations on ordinals. The Bulletin of Symbolic Logic, 11(3 (Sep., 2005)):377–397, 2005.

[Koe09] Peter Koellner. On Reflection Principles. Annals of Pure and Applied Logic, 157(2-3):206–219, 2009.

[Koe10] Peter Koellner. Strong logics of first and second order. Bulletin of Symbolic Logic, 16(1):1–36, 03 2010.

[Kre61] George Kreisel. Set theoretic problems suggested by the notion of potential totality. Infinitistic Methods, pages 103–140, 1961.

[Kre71] Georg Kreisel. Some Reasons for Generalizing Recursion Theory. In R.O. Gandy and C.E.M. Yates, editors, Logic Colloquium '69, volume 61 of Studies in Logic and the Foundations of Mathematics, pages 139–198. North-Holland, Amsterdam, 1971.

[KS65] G. Kreisel and Gerald E. Sacks. Metarecursive sets. Journal of Symbolic Logic, 30(3):318–338, 09 1965.

[KS09] Peter Koepke and Benjamin Seyfferth. Ordinal machines and admissible recursion theory. Annals of Pure and Applied Logic, 160(3):310 – 318, 2009.

[Kun80] Kenneth Kunen. Set theory : an introduction to independence proofs. Studies in logic and the foundations of mathematics. Elsevier science, Amsterdam, Lausanne, New York, 1980. Fifth impression, 1992.

[Kur22] Casimir Kuratowski. Une méthode d'élimination des nombres transfinis des raisonnements mathématiques. Fundamenta Mathematicae, 3(1):76–108, 1922.

[Lé60] Azriel Lévy. Axiom schemata of strong infinity in axiomatic set theory. Pacific Journal of Mathematics, 10(1):223–238, 1960.

[Lan99] Elaine Landry. Category theory as a framework for mathematical structuralism. The 1998 Annual Proceedings of the Canadian Society for the History and Philosophy of Mathematics, pages 133–142, 1999.

[Lav99] Shaughan Lavine. Skolem was Wrong. Unpublished. Dated June, 1999.

[Lin08] Øystein Linnebo. Structuralism and the Notion of Dependence. The Philosophical Quarterly, 58(230):59–79, 2008.

[Mar70] Donald A. Martin. Review of [Qui69]. The Journal of Philosophy, 67(4):111–114, 1970.

[Mat01] A. R. D. Mathias. Slim models of zermelo set theory. J. Symb. Log., 66(2):487–496, 2001.

[Pot93] M. D. Potter. Iterative set theory. The Philosophical Quarterly, 43(171):pp. 178–193, 1993.

[Pot04] Michael Potter. Set theory and its philosophy, 2004.

[Qui69] W.V. Quine. Set Theory and Its Logic. Revised edition. Harvard University Press, Cambridge, 1969.

[Rei74] W. N. Reinhardt. Remarks on Reflection Principles, Large Cardinals, and Elementary Embeddings. In Thomas Jech, editor, Axiomatic Set Theory, pages 189–205. American Mathematical Society, Providence, 1974.

[Sac90] G.E. Sacks. Higher Recursion Theory. Perspectives in Mathematical Logic. Springer-Verlag GmbH, 1990.

[Sie05] Wilfried Sieg. Church without dogma: Axioms for computability. 2005.

[Soa87] Robert I. Soare. Recursively Enumerable Sets and Degrees. Perspectives in Mathematical Logic. Springer, Berlin, 1987.

[Tap08] Jamie Tappenden. Mathematical concepts and definitions. In Paolo Mancosu, editor, The Philosophy of Mathematical Practice, pages 256–275. OUP Oxford, 2008.

[Tur36] Alan M. Turing. On computable numbers, with an application to the entscheidungsproblem. Proceedings of the London Mathematical Society, 2(42):230–265, 1936.