

ATTEMPTS TO RETHINK LOGIC

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The period between Kant and Frege is widely held to be an inactive time in the history of logic, especially when compared to the periods that preceded and succeeded it. By the late eighteenth century, the rich and suggestive exploratory work of Leibniz had led to writings in symbolic logic by Lambert and Ploucquet.¹ But after Lambert this tradition effectively ended, and some of its innovations had to be rediscovered independently later in the century. Venn characterized the period between Lambert and Boole as “almost a blank in the history of the subject” and confessed an “uneasy suspicion” that a chief cause was the “disastrous effect on logical method” wrought by Kant’s philosophy.² De Morgan began his work in symbolic logic “facing Kant’s assertion that logic neither has improved since the time of Aristotle, nor of its own nature can improve.”³

De Morgan soon discovered, however, that the leading logician in Britain at the time, William Hamilton, had himself been teaching that the traditional logic was “perverted and erroneous in form.”⁴ In Germany, Maimon argued

¹ J. H. Lambert, *Sechs Versuche einer Zeichenkunst in der Vernunftlehre*, in *Logische und philosophische Abhandlungen*, ed. J. Bernoulli, vol. 1 (Berlin, 1782). G. Ploucquet, *Sammlung der Schriften welche den logischen Calcul Herrn Professor Ploucquet’s betreffen, mit neuen Zusätzen* (Frankfurt, 1766).

² John Venn, *Symbolic Logic*, 2nd ed. (London: Macmillan, 1894), xxxvii. (Original edition, 1881).

³ Augustus De Morgan, “On the Syllogism III,” reprinted in *On the Syllogism and Other Writings*, ed. P. Heath (New Haven, Conn.: Yale University Press, 1966), 74. (Original edition, 1858.) De Morgan is referring to Immanuel Kant, *Critique of Pure Reason*, eds. and trans. Paul Guyer and Allen Wood (Cambridge: Cambridge University Press, 1998), Bviii. For the *Critique of Pure Reason*, I follow the common practice of citing the original page numbers in the first (A) or second (B) edition of 1781 and 1787. Citations of works of Kant besides the *Critique of Pure Reason* are according to the German Academy (Ak) edition pagination: *Gesammelte Schriften*, ed. Königlich Preussische (later Deutsche und Berlin-Brandenburgische) Akademie der Wissenschaften, 29 vols. (Berlin: G. Reimer [later Walter de Gruyter], 1900–). Passages from Kant’s *Logic* (edited by Kant’s student Jäsche and published under Kant’s name in 1800) are also cited by paragraph number (§) when appropriate. I use the translation in *Lectures on Logic*, ed. and trans. J. Michael Young (Cambridge: Cambridge University Press, 1992).

⁴ De Morgan, “Syllogism III,” 75. Cf. William Hamilton, *Lectures on Logic*, eds. H. L. Mansel and John Veitch, 3rd ed., 2 vols. (London: Blackwood, 1874), 2:251.

that Kant treated logic as complete only because he omitted the most important part of critique – a critique of logic itself.⁵ Hegel, less interested in formal logic than Maimon, concurs that “if logic has not undergone any change since Aristotle, . . . then surely the conclusion which should be drawn is that it is all the more in need of a total reconstruction.”⁶ On Hegel’s reconstruction, logic “coincides with metaphysics.”⁷ Fries argued that Kant thought logic complete only because he neglected “anthropological logic,” a branch of empirical psychology that provides a theory of the capacities humans employ in thinking and a basis for the meager formal content given in “demonstrative” logic.⁸ Trendelenburg later argued that the logic contained in Kant’s *Logic* is not Aristotle’s logic at all, but a *corruption* of it, since Aristotelian logic has metaphysical implications that Kant rejects.⁹

Indeed, one would be hard pressed to find a single nineteenth-century logician who agrees with Kant’s notorious claim. However, this great expansion of logic – as some logical works branched out into metaphysics, epistemology, philosophy of science, and psychology, while others introduced new symbolic techniques and representations – threatened to leave logicians with little common ground except for their rejection of Kant’s conservatism. Robert Adamson, in his survey of logical history for the *Encyclopedia Britannica*, writes of nineteenth-century logical works that “in tone, in method, in aim, in fundamental principles, in extent of field, they diverge so widely as to appear, not so many expositions of the same science, but so many different sciences.”¹⁰ Many historians of logic have understandably chosen to circumvent this problem by ignoring many of the logical works that were the most widely read and

⁵ Salomon Maimon, *Versuch einer neuen Logik oder Theorie des Denkens: Nebst angehängten Briefen des Philaletes an Aenesidemus* (Berlin: Ernst Felisch, 1794), 404ff. Partially translated by George di Giovanni as *Essay towards a New Logic or Theory of Thought, Together with Letters of Philaletes to Aenesidemus*, in *Between Kant and Hegel*, rev. ed. (Indianapolis: Hackett, 2000). Citations are from the pagination of the original German edition, which are reproduced in the English translation.

⁶ G. W. F. Hegel, *Science of Logic*, trans. A. V. Miller (London: George Allen & Unwin, 1969), 51. German edition: *Gesammelte Werke* (Hamburg: Felix Meiner, 1968–), 21:35. (Original edition, 1812–16, first volume revised in 1832.) I cite from the now-standard German edition, which contains the 1832 edition of “The Doctrine of Being,” the 1813 edition of “The Doctrine of Essence,” and the 1816 edition of “The Doctrine of the Concept” in vols. 21, 11, and 12, respectively.

⁷ G. W. F. Hegel, *The Encyclopedia Logic*, pt. 1 of the *Encyclopedia of Philosophical Sciences*, trans. T. F. Geraets, W. A. Suchting, and H. S. Harris (Indianapolis: Hackett, 1991), §24. *Gesammelte Werke* (Hamburg: Felix Meiner, 1968–), 20: §24. (Original edition, 1817; second edition, 1830.) I cite paragraph numbers (§) throughout.

⁸ Jakob Friedrich Fries, *System der Logik*, 3rd ed. (Heidelberg, 1837), 4–5. (Original edition, 1811.)

⁹ Adolf Trendelenburg, *Logische Untersuchungen*, 3rd ed., 2 vols. (Leipzig: S. Hirzel, 1870), 1:32–3. (Original edition, 1840.)

¹⁰ Robert Adamson, *A Short History of Logic* (London: W. Blackwood, 1911), 20. (Original edition, 1882.)

discussed during the period – the works of Hegel, Trendelenburg, Hamilton, Mill, Lotze, and Sigwart, for example.

The present article, however, aims to be a history of “logic” in the multifaceted ways in which this term was understood between Kant and Frege (though the history of inductive logic – overlapping with the mathematical theory of probabilities and with questions about scientific methodology – falls outside the purview of this article). There are at least two reasons for this wide perspective. First, the diversity of approaches to logic was accompanied by a continuous debate in the philosophy of logic over the nature, extent, and proper method in logic. Second, the various logical traditions that coexisted in the period – though at times isolated from one another – came to cross-pollinate with one another in important ways. The first three sections of the article trace out the evolving conceptions of logic in Germany and Britain. The last three address the century’s most significant debates over the nature of concepts, judgments and inferences, and logical symbolism.

KANTIAN AND POST-KANTIAN LOGICS

Surprisingly, Kant was widely held in the nineteenth century to have been a logical innovator. In 1912 Wilhelm Windelband wrote: “a century and a half ago, [logic] . . . stood as a well-built edifice firmly based on the Aristotelian foundation. . . . But, as is well-known, this state of things was entirely changed by Kant.”¹¹ Kant’s significance played itself out in two opposed directions: first, in his novel characterization of logic as formal; and, second, in the new conceptions of logic advocated by those post-Kantian philosophers who drew on Kant’s *transcendental* logic to attack Kant’s own narrower conception of the scope of logic.

Though today the idea that logic is formal seems traditional or even definitional, nineteenth-century logicians considered the idea to be a Kantian innovation. Trendelenburg summarized the recent history:¹²

Christian Wolff is still of the view that the grounds of logic derive from ontology and psychology and that logic precedes them only in the order in which the sciences are

¹¹ Wilhelm Windelband, *Theories in Logic*, trans. B. Ethel Meyer (New York: Citadel Press, 1961), 1. (Original edition, 1912.)

¹² Contemporary Kant interpreters do not agree on whether Kant’s thesis that logic is formal was in fact novel. Some agree with Trendelenburg that Kant’s thesis was an innovative doctrine that depended crucially on other distinctive features of Kant’s philosophy; others think that Kant was reviving or modifying a traditional, Scholastic conception. (For an excellent recent paper that defends Trendelenburg’s view, see John MacFarlane, “Frege, Kant, and the Logic in Logicism,” *Philosophical Review* 111 [2000]: 25–65.) However, irrespective of the accuracy of this interpretive claim, the point remains that the thesis that logic is formal was considered by nineteenth-century

studied. For the first time in Kant's critical philosophy, in which the distinction of matter and form is robustly conceived, *formal logic* clearly emerges and actually stands and falls with Kant.¹³

General logic for Kant contains the "absolutely necessary rules of thinking, without which no use of the understanding takes place."¹⁴ The understanding – which Kant distinguishes from "sensibility" – is the faculty of "thinking," or "cognition through concepts."¹⁵ Unlike Wolff, Kant claims a *pure* logic "has no empirical principles, thus it draws nothing from psychology."¹⁶ The principles of psychology tell how we *do* think; the principles of pure general logic, how we *ought* to think.¹⁷ The principles of logic do not of themselves imply metaphysical principles; Kant rejects Wolff and Baumgarten's proof of the principle of sufficient reason from the principle of contradiction.¹⁸ Though logic is a canon, a set of rules, it is not an organon, a method for expanding our knowledge.¹⁹

For Kant, pure general logic neither presupposes nor of itself implies principles of any other science because it is *formal*. "General logic abstracts . . . from all content of cognition, i.e., from any relation of it to the object, and considers only the logical form in the relation of cognitions to one another."²⁰ In its treatment of concepts, formal logic takes no heed of the particular marks that a given concept contains, nor of the particular objects that are contained under it. In its treatment of judgments, formal logic attends merely to the different ways in which one concept can be contained in or under one another. (So in a judgment like "All whales are mammals," the word "all" and the copula "is" do not represent concepts, but express the *form* of the judgment, the particular way in which a thinker combines the concepts *whale* and *mammal*.)

logicians to be a Kantian innovation dependent on other parts of Kant's philosophy. Besides Trendelenburg, see also Maimon, *Logik*, xx; William Hamilton, "Recent Publications on Logical Sciences," reprinted in his *Discussions on Philosophy and Literature, Education and University Reform* (New York: Harper, 1861), 145; J. S. Mill, *An Examination of Sir William Hamilton's Philosophy*, reprinted as vol. 9 of *Collected Works of John Stuart Mill*, ed. J. M. Robson (Toronto: University of Toronto Press, 1979), 355; De Morgan, "Syllogism III," 76; and Henry Mansel, *Prolegomena Logica: An Inquiry into the Psychological Character of Logical Processes*, 2nd ed. (Boston, 1860), iv–v. Among these logicians, Hamilton was unique in recognizing the affinity between Kant's conception of logic and Scholastic notions.

¹³ Trendelenburg, *Untersuchungen*, I, 15. In this passage, Trendelenburg cites Christian Wolff, *Philosophia rationalis sive Logica* (Frankfurt, 1728), §88–9.

¹⁴ Kant, *Critique of Pure Reason*, A52/B76.

¹⁵ Kant, *Critique of Pure Reason*, A50/B74; Kant, Ak 9:91.

¹⁶ Kant, *Critique of Pure Reason*, A54/B78.

¹⁷ Kant, Ak 9:14.

¹⁸ Kant, Ak 4:270. See Christian Wolff, *Philosophia prima sive ontologia* (Frankfurt, 1730), §70. Alexander Gottlieb Baumgarten, *Metaphysica*, 3rd ed. (Halle: Hammerde, 1757).

¹⁹ Kant, Ak 9:13.

²⁰ Kant, *Critique of Pure Reason*, A55/B79.

The generality of logic requires this kind of formality because Kant, as an essential part of his critique of dogmatic metaphysicians such as Leibniz and Wolff, distinguishes mere thinking from cognizing (or knowing).²¹ Kant argues against traditional metaphysics that, since we can have no intuition of noumena, we cannot have cognitions or knowledge of them. But we can coherently *think* noumena.²² This kind of thinking is necessary for *moral faith*, where the subject is not an object of intuition, but, for example, the divine being as moral lawgiver and just judge. Thus, formal logic, which abstracts from all content of cognition, makes it possible for us coherently to conceive of God and things in themselves.

The thesis of the formality of logic, then, is intertwined with some of the most controversial aspects of the critical philosophy: the distinction between sensibility and understanding, appearances and things in themselves. Once these Kantian “dualisms” came under severe criticism, post-Kantian philosophers also began to reject the possibility of an independent formal logic.²³ Hegel, for example, begins his *Science of Logic* with a polemic against Kant’s conception of formal logic: if there are no unknowable things in themselves, then the rules of thinking are rules for thinking an object, and the principles of logic become the first principles of ontology.²⁴

Further, Kant’s insistence that the principles of logic are not drawn from psychology or metaphysics leaves open a series of epistemological questions. How then do we know the principle of contradiction? How do we know that there are precisely twelve logical forms of judgment? Or that some figures of the syllogism are valid and others not? Many agreed with Hegel that Kant’s answers had “no other justification than that we *find* such species *already* to

²¹ Kant, *Critique of Pure Reason*, B146.

²² Kant, *Critique of Pure Reason*, B166n.

²³ A classic attack on Kant’s distinction between sensibility and understanding is Salomon Maimon, *Versuch über die Transzendentalphilosophie*, reprinted in *Gesammelte Werke*, ed. Valerio Verra, vol. 2, (Hildesheim: Olms, 1965–76), 63–4. A classic attack on Kant’s distinction between appearances and things in themselves is F. H. Jacobi, *David Hume über den Glauben, oder Idealismus und Realismus: Ein Gespräch* (Breslau: Gottlieb Löwe, 1787). I have emphasized that Kant defends the coherence of the doctrine of unknowable things in themselves by distinguishing between thinking and knowing – where thinking, unlike knowing, does not require the joint operation of sensibility and understanding. Formal logic, by providing rules for the use of the understanding and abstracting from all content provided by sensibility, makes room for the idea that we can coherently think of things in themselves. Now, the fact that Kant defends the coherence of his more controversial doctrines by appealing to the formality of logic does not yet imply that an attack on Kantian “dualisms” need also undermine the thesis that logic is formal. But, as we will see, many post-Kantian philosophers thought that an attack on Kant’s distinctions would also undermine the formality thesis – or at least they thought that such an attack would leave the formality of logic unmotivated.

²⁴ Hegel, *Science of Logic*, 43–8 (*Werke* 21:28–32).

hand and they present themselves *empirically*.”²⁵ Kant’s unreflective procedure endangers both the a priori *purity* and the *certainty* of logic.²⁶

Though Kant says only that “the labors of the logicians were ready to hand,”²⁷ his successors were quick to propose novel answers to these questions. Fries appealed to introspective psychology, and he reproved Kant for overstating the independence of “demonstrative logic” from anthropology.²⁸ Others grounded formal logic in what Kant called “transcendental logic.” Transcendental logic contains the rules of a priori thinking.²⁹ Since all use of the understanding, inasmuch as it is cognizing an object, requires a priori concepts (the categories), transcendental logic then expounds also “the principles without which no object can be thought at all.”³⁰ Reinhold argued – against Kant’s “Metaphysical Deduction” of the categories from the forms of judgments – that the principles of pure general logic should be derived from a transcendental principle (such as his own principle of consciousness).³¹ Moreover, logic can only be a science if it is systematic, and this systematicity (on Reinhold’s view) requires that logic be derived from an indemonstrable first principle.³²

Maimon’s 1794 *Versuch einer neuen Logik oder Theorie des Denkens*, which contains both an extended discussion of formal logic and an extended transcendental logic, partially carries out Reinhold’s program. There are two highest principles: the principle of contradiction (which is the highest principle of all analytic judgments) and Maimon’s own “principle of determinability” (which is the highest principle of all synthetic judgments).³³ Since formal logic presupposes transcendental logic,³⁴ Maimon defines the various forms of judgment (such as affirmative and negative) using transcendental concepts (such as reality

²⁵ Hegel, *Science of Logic*, 613 (Werke 12:43).

²⁶ See also J. G. Fichte, “First Introduction to the Science of Knowledge,” in *The Science of Knowledge*, trans. Peter Heath and John Lachs (Cambridge: Cambridge University Press, 1982), 1:442. See *Johann Gottlieb Fichtes sämtliche Werke*, ed. I. H. Fichte, vol. 1 (Berlin: Veit, 1845–6), reprinted as *Fichtes Werke* (Berlin: Walter de Gruyter, 1971). (Original edition of “First Introduction,” 1797.) Citations are by volume and page number from I. H. Fichte’s edition, which are reproduced in the margins of the English translations. K. L. Reinhold, *The Foundation of Philosophical Knowledge*, trans. George di Giovanni in *Between Kant and Hegel*, rev. ed. (Indianapolis: Hackett, 2000), 119. Citations follow the pagination of the original 1794 edition, *Über das Fundament des philosophischen Wissens* (Jena, 1794), which are reproduced in the margins of the English edition.

²⁷ Kant, Ak 4:323.

²⁸ Fries, *Logik*, 5.

²⁹ Kant, *Critique of Pure Reason*, A57/B81.

³⁰ Kant, *Critique of Pure Reason*, A62/B87.

³¹ Reinhold, *Foundation*, 118–21.

³² See also J. G. Fichte, “Concerning the Concept of the *Wissenschaftslehre*,” trans. Daniel Breazeale in *Fichte: Early Philosophical Writings* (Ithaca, N.Y.: Cornell University Press, 1988), 1:41–2. (Original edition, 1794.)

³³ Maimon, *Versuch einer neuen Logik oder Theorie des Denkens*, 19–20.

³⁴ Maimon, *Versuch einer neuen Logik oder Theorie des Denkens*, xx–xxii.

and negation). He proves various features of syllogisms (such as that the conclusion of a valid syllogism is affirmative iff both its premises are) using the transcendental principle of determinability.³⁵

Hegel's *Science of Logic* is surely the most ambitious and influential of the logical works that include both formal and transcendental material.³⁶ However, Hegel cites as a chief inspiration – not Maimon, but – Fichte. For Hegel, Fichte's philosophy “reminded us that the *thought-determinations* must be exhibited in their *necessity*, and that it is essential for them to be *deduced*.”³⁷ In Fichte's *Wissenschaftslehre*, the whole system of necessary representations is deduced from a single fundamental and indemonstrable principle.³⁸ In his 1794 book, *Foundations of the Entire Science of Knowledge*, Fichte derives from the first principle “I am I” not only the category of reality, but also the logical law “ $A = A$ ”; in subsequent stages he derives the category of negation, the logical law “ $\sim A$ is not equal to A ,” and finally even the various logical forms of judgment. Kant's reaction, given in his 1799 “Open Letter on Fichte's *Wissenschaftslehre*,” is unsurprising: Fichte has confused the proper domain of logic with metaphysics.³⁹ Later, Fichte follows the project of the *Wissenschaftslehre* to its logical conclusion: transcendental logic “destroys” the common logic in its foundations, and it is necessary to refute (in Kant's name) the very possibility of formal logic.⁴⁰

Hegel turned Kant's criticism of Fichte on its head: were the critical philosophy consistently thought out, logic and metaphysics would in fact coincide. For Kant, the understanding can combine or synthesize a sensible manifold, but it cannot itself produce the manifold. For Hegel, however, there can be an *absolute* synthesis, in which thinking itself provides contentful concepts independently of sensibility.⁴¹ Kant had argued that if the limitations of thinking are disregarded, reason falls into illusion. In a surprising twist, Hegel uses this dialectical nature of pure reason to make possible his own non-Kantian doctrine of synthesis. Generalizing Kant's antinomies to all concepts, Hegel argues that any pure concept, when thought through, leads to its opposite.⁴² This back-and-forth transition from a concept to its opposite – which Hegel

³⁵ Maimon, *Versuch einer neuen Logik oder Theorie des Denkens*, 94–5.

³⁶ Hegel, *Science of Logic*, 62 (Werke 21:46).

³⁷ Hegel, *Encyclopedia Logic*, §42.

³⁸ Fichte, “First Introduction,” 1:445–6.

³⁹ Kant, Ak 12:370–1.

⁴⁰ J. G. Fichte, “Ueber das Verhältniß der Logik zur Philosophie oder transscendentale Logik,” reprinted in *Fichtes Werke*, 9:111–12. This text is from a lecture course delivered in 1812.

⁴¹ G. W. F. Hegel, *Faith and Knowledge*, trans. W. Cerf and H. S. Harris (Albany, N.Y.: SUNY Press, 1977), 72 (Werke 4:328). (Original edition, 1802.)

⁴² Hegel, *Encyclopedia Logic*, §48.

calls the “dialectical moment” – can itself be synthesized into a new unitary concept – which Hegel calls the “speculative moment,” and the process can be repeated.⁴³ Similarly, the most immediate judgment, the positive judgment (e.g., “the rose is red”), asserts that the individual is a universal. But since the rose is more than red, and the universal *red* applies to more than the rose, the individual is not the universal, and we arrive at the *negative* judgment.⁴⁴ Hegel iterates this procedure until he arrives at a complete system of categories, forms of judgment, logical laws, and forms of the syllogism. Moreover, by beginning with the absolutely indeterminate and abstract thought of *being*,⁴⁵ he has rejected Reinhold’s demand that logic begin with a first principle: Hegel’s *Logic* is “preceded by . . . total presuppositionlessness.”⁴⁶

THE REVIVAL OF LOGIC IN BRITAIN

The turnaround in the fortunes of logic in Britain was by near consensus attributed to Whately’s 1826 *Elements of Logic*.⁴⁷ A work that well illustrates the prevailing view of logic in the Anglophone world before Whately is Harvard Professor of Logic Levi Hedge’s *Elements of Logick*. The purpose of logic, Hedge claims, is to direct the intellectual powers in the investigation and communication of truths.⁴⁸ This means that a logical treatise must trace the progress of knowledge from simple perceptions to the highest discoveries of reasoning. The work thus reads more like Locke’s *Essay* than Kant’s *Logic* – it draws heavily not only on Locke, but also on Reid and on Hume’s laws of the associations of ideas. Syllogistic, however, is discussed only in a footnote – since syllogistic is “of no use in helping us to the discovery of new truths.”⁴⁹ Hedge here cites Locke’s *Essay*,⁵⁰ where Locke argued that syllogistic is not necessary for reasoning well – “God has not been so sparing to Men to make them barely two-legged Creatures, and left it to *Aristotle* to make them Rational.” Syllogisms, Locke claims, are of no use in the discovery of new truths or the finding of new

⁴³ Hegel, *Encyclopedia Logic*, §§81–2.

⁴⁴ Hegel, *Science of Logic*, 594, 632ff. (Werke 12:27, 61ff.).

⁴⁵ Hegel, *Science of Logic*, 70 (Werke 21:58–9).

⁴⁶ Hegel, *Encyclopedia Logic*, §78.

⁴⁷ See De Morgan, “Logic,” reprinted in *On the Syllogism*, 247. (De Morgan’s essay first appeared in 1860.) Hamilton, “Recent Publications,” 128. (Original edition, 1833.) John Stuart Mill, *A System of Logic, Ratiocinative and Inductive*, reprinted as vols. 7 and 8 of *Collected Works of John Stuart Mill*, ed. John M. Robson (Toronto: University of Toronto Press, 1973), 7:cxiv. (Original edition, 1843.)

⁴⁸ Levi Hedge, *Elements of Logick* (Cambridge, Mass.: Harvard University Press, 1816), 13.

⁴⁹ Hedge, *Logick*, 152, 148.

⁵⁰ John Locke, *An Essay Concerning Human Understanding*, ed. P. H. Nidditch, based on 4th ed. (New York: Oxford University Press, 1975), IV.xvii.4.

proofs; indeed, they are inferior in this respect to simply arranging ideas “in a simple and plain order.”

For Whately, Locke’s objection that logic is unserviceable in the discovery of the truth misses the mark, because it assumes a mistaken view of logic.⁵¹ The chief error of the “schoolmen” was the unrealizable expectation they raised: that logic would be an art that furnishes the sole instrument for the discovery of truth, that the syllogism would be an engine for the investigation of nature.⁵² Fundamentally, logic is a science and not an art.⁵³ Putting an argument into syllogistic form need not add to the certainty of the inference, any more than natural laws make it more certain that heavy objects fall. Indeed, Aristotle’s *dictum de omni et nullo* is like a natural law: it provides an *account* of the correctness of an argument; it shows us the one general principle according to which takes place every individual case of correct reasoning. A logician’s goal then is to show that all correct reasoning is conducted according to one general principle – Aristotle’s *dictum* – and is an instance of the same mental process – syllogistic.⁵⁴

In Hamilton’s wide-ranging and erudite review of Whately’s *Elements*, he acknowledged that Whately’s chief service was to correct mistakes about the nature of logic, but he excoriated his fellow Anglophones for their ignorance of historical texts and contemporary German logics. Indeed, we can more adequately purify logic of intrusions from psychology and metaphysics and more convincingly disabuse ourselves of the conviction that logic is an “instrument of scientific discovery” by accepting Kant’s idea that logic is *formal*.⁵⁵ Hamilton’s lectures on logic, delivered in 1837–8 using the German Kantian logics written by Krug and Esser,⁵⁶ thus introduced into Britain the Kantian idea that logic is formal.⁵⁷ For him, the form of thought is the kind and manner of thinking an object⁵⁸ or the relation of the subject to the object.⁵⁹ He distinguishes logic from psychology (against Whately) as the science of the *product*, not the *process*, of thinking. Since the forms of thinking studied by logic are *necessary*, there must be *laws* of thought: the principles of identity, contradiction, and excluded middle).⁶⁰ He distinguishes physical laws

⁵¹ Richard Whately, *Elements of Logic*, 9th ed. (London: Longmans, Green, Reader, & Dyer, 1866), 15. (Original edition, 1826.)

⁵² Whately, *Elements of Logic*, viii, 4, 5.

⁵³ Whately, *Elements of Logic*, 1.

⁵⁴ Whately, *Elements of Logic*, 75.

⁵⁵ Hamilton, “Recent Publications,” 139.

⁵⁶ Wilhelm Traugott Krug, *Denklehre oder Logik* (Königsberg: Goebbels & Unzer, 1806). Jakob Esser, *System der Logik*, 2nd ed. (Münster, 1830). (Original edition, 1823.)

⁵⁷ Hamilton, *Logic*. I cite from the 1874 3rd ed. (Original edition, 1860.)

⁵⁸ Hamilton, *Logic*, 1:13.

⁵⁹ Hamilton, *Logic*, 1:73.

⁶⁰ Hamilton, *Logic*, 1:17, 2:246.

from “formal laws of thought,” which thinkers *ought* to – though they do not always – follow.⁶¹

Mill later severely (and justly) criticized Hamilton for failing to characterize the distinction between the matter and form of thinking adequately. Mill argued that it is impossible to take over Kant’s matter/form distinction without also taking on all of Kant’s transcendental idealism.⁶² Mansel tries to clarify the distinction between matter and form by arguing that the form of thinking is expressed in analytic judgments. He claims (as neither Hamilton nor Kant himself had done explicitly) that the three laws of thought are themselves analytic judgments and that the entire content of logic is derivable from these three laws.⁶³ Moreover, Mansel further departs from Kant and Hamilton by restricting the task of logic to characterizing the form and laws of *only* analytic judgments.⁶⁴

In his 1828 review, Mill criticized Whately for concluding that inductive logic – that is, the rules for the investigation and discovery of truth – could never be put into a form as systematic and scientific as syllogistic.⁶⁵ Mill’s *System of Logic, Ratiocinative and Inductive*, which was centered around Mill’s famous five canons of experimental inquiry, aimed to do precisely what Whately thought impossible. The work, which included material we would now describe as philosophy of science, went through eight editions and became widely used in colleges throughout nineteenth-century Britain. Logic for Mill is the science as well as the art of reasoning;⁶⁶ it concerns the operations of the understanding in giving proofs and estimating evidence.⁶⁷ Mill argued that in fact all reasoning is inductive.⁶⁸ There is an inconsistency, Mill alleges, in thinking that the conclusion of a syllogism (e.g., “Socrates is mortal”) is known on the basis of the premises (e.g., “All humans are mortal” and “Socrates is human”), while also admitting that the syllogism is vicious if the conclusion is not already asserted

⁶¹ Hamilton, *Logic*, 1:78. Hamilton unfortunately muddies the distinction between the two kinds of laws by lining it up with the Kantian distinction between the doctrine of elements and the doctrine of method (*Logic*, 1:64).

⁶² Mill, *Examination*, 355. (Original edition, 1865.)

⁶³ Mansel, *Prolegomena*, 159. (Original edition, 1851.) A similar position had been defended earlier by Fries, *Logik*, §40. Mansel almost certainly got the idea from Fries.

⁶⁴ Mansel, *Prolegomena*, 202. Although Kant did say that the truth of an analytic judgment could be derived entirely from logical laws (see, e.g., *Critique of Pure Reason*, A151/B190–1), he always thought that logic considered the forms of all judgments whatsoever, whether analytic or synthetic.

⁶⁵ John Stuart Mill, “Whately’s Elements of Logic,” reprinted in vol. 11 of *Collected Works of John Stuart Mill*, ed. J. M. Robson (Toronto: University of Toronto Press, 1978). Cf. Whately, *Elements*, 168; Mill, *Logic*, cxii.

⁶⁶ Mill, *Logic*, 5.

⁶⁷ Mill, *Logic*, 12.

⁶⁸ Mill, *Logic*, 202.

in the premises.⁶⁹ Mill's solution to this paradox is that the syllogistic inference is only apparent: the real inference is the induction from the particular facts about the mortality of particular individuals to the mortality of Socrates. The inference is then actually finished when we assert, "All men are mortal."⁷⁰

The debate over whether logic is an art and the study of logic useful for reasoning dovetailed with concurrent debates over curricular reform at Oxford. By 1830, Oxford was the only British institution of higher learning where the study of logic had survived.⁷¹ Some, such as Whewell, advocated making its study elective, allowing students to train their reasoning by taking a course on Euclid's *Elements*.⁷² Hamilton opposed this proposal,⁷³ as did the young mathematician Augustus De Morgan, who thought that the study of syllogistic facilitates a student's understanding of geometrical proofs.⁷⁴ Indeed, De Morgan's first foray into logical research occurred in a mathematical textbook, where, in a chapter instructing his students on putting Euclidean proofs into syllogistic form, he noticed that some proofs require treating "is equal to" as a copula distinct from "is," though obeying all of the same rules.⁷⁵

These reflections on mathematical pedagogy led eventually to De Morgan's logical innovations. In his *Formal Logic*,⁷⁶ De Morgan noted that the rules of the syllogism work for copulae other than "is" – as long as they have the formal properties of transitivity, reflexivity, and what De Morgan calls "contrariety."⁷⁷ Transitivity is the common form, and what distinguishes "is" from "is equal to" is their matter. This generalization of the copula culminated in De Morgan's paper "On the Syllogism IV," the first systematic study of the logic of relations.⁷⁸ He considers propositions of the form "*A.LB*" ("*A* is one of the *Ls* of *B*"), where "*L*" denotes any relation of subject to predicate. He thinks of "*A*" as the subject term, "*B*" as the predicate term, and "*.L*" as the relational expression that functions as the copula connecting subject and predicate.⁷⁹

⁶⁹ Mill, *Logic*, 185.

⁷⁰ Mill, *Logic*, 186–7.

⁷¹ Hamilton, "Recent Publications," 124. De Morgan, "On the Methods of Teaching the Elements of Geometry," *Quarterly Journal of Education* 6 (1833): 251.

⁷² W. Whewell, *Thoughts on the Study of Mathematics as Part of a Liberal Education* (Cambridge: Deighton, 1835).

⁷³ Hamilton, "On the Study of Mathematics, as an Exercise of Mind," reprinted in *Discussions on Philosophy and Literature, Education and University Reform*. (Original edition, 1836.)

⁷⁴ De Morgan, "Methods of Teaching," 238–9.

⁷⁵ Augustus De Morgan, *On the Study and Difficulties of Mathematics* (London: Society for the Diffusion of Useful Knowledge, 1831).

⁷⁶ Augustus De Morgan, *Formal Logic* (London: Taylor & Walton, 1847).

⁷⁷ De Morgan, *Formal Logic*, 57–9.

⁷⁸ De Morgan, "On the Syllogism IV," reprinted in *On the Syllogism*. (Original edition, 1860.)

⁷⁹ The two dots preceding "*L*" indicate that the statement is affirmative; an odd number of dots indicates that the proposition is negative.

Thus, if we let “ L ” mean “loves,” then “ $A..LB$ ” would mean that “ A is one of the lovers of B ” – or just “ A loves B .” He symbolized contraries using lower-case letters: “ $A..lB$ ” means “ A is one of the nonlovers of B ” or “ A does not love B .” Inverse relations are symbolized using the familiar algebraic expression: “ $A..L^{-1}B$ ” means “ A is loved by B .” Importantly, De Morgan also considered compound relations, or what we would now call “relative products”: “ $A..LPB$ ” means “ A is a lover of a parent of B .” De Morgan recognized, moreover, that reasoning with compound relations required some simple quantificational distinctions. We symbolize “ A is a lover of every parent of B ” by adding an accent: “ $A..LP'B$.” De Morgan was thus able to state some basic facts and prove some theorems about compound relations. For instance, the contrary of LP is lP' and the converse of the contrary of LP is $p^{-1}L^{-1}$.⁸⁰

De Morgan recognized that calling features of the copula “is” *material* departed from the Kantian view that the copula is part of the form of a judgment.⁸¹ De Morgan, however, thought that the logician’s matter/form distinction could be clarified by the mathematician’s notion of form.⁸² From the mathematician’s practice we learn two things. First, the form/matter distinction is relative to one’s level of abstraction: the algebraist’s $x + y$ is formal with respect to $4 + 3$, but $x + y$ as an operation on numbers is distinguished only materially from the similar operations done on vectors or differential operators.⁸³ Second, the form of thinking is best understood on analogy with the principle of a *machine* in operation.⁸⁴

In thinking of mathematics as a mechanism, De Morgan is characterizing mathematics as fundamentally a matter of applying operations to symbols according to laws of their combinations. Here De Morgan is drawing on work done by his fellow British algebraists. (In fact, De Morgan’s logical work is the confluence of three independent intellectual currents: the debate raging from Locke to Whately over the value of syllogistic, the German debate – imported by Hamilton – over Kant’s matter/form distinction,⁸⁵ and the mathematical debate

⁸⁰ Spelled out a bit (and leaving off the quotation marks for readability): that LP is the contrary of lP' means that $A..LPB$ is false iff $A..lP'B$ is true. That is, A does not love any of B ’s parents iff A is a nonlover of every parent of B . That the converse of the contrary of LP is $p^{-1}L^{-1}$ means that $A..LPB$ is false iff $B..p^{-1}L^{-1}A$ is true. That is, A does not love any of B ’s parents iff B is not the child of anyone A loves.

⁸¹ De Morgan, “On the Syllogism II,” reprinted in *On the Syllogism*, 57–8. (Original edition, 1850.)

⁸² De Morgan, “Syllogism III,” 78.

⁸³ De Morgan, “Logic,” 248; “Syllogism III,” 78.

⁸⁴ De Morgan, “Syllogism III,” 75.

⁸⁵ We saw earlier that Mill argued that one could not maintain that logic is “formal” unless one were willing to take on Kant’s matter/form distinction – and therefore also Kant’s *transcendental idealism*. De Morgan, on the other hand, thought that introducing the *mathematician’s* notion of “form” would allow logicians to capture what is correct in Kant’s idea that logic is formal, but without having to take on the rest of the baggage of Kantianism. However, the effect of making

centered at Cambridge over the justification of certain algebraic techniques.) As a rival to Newton's geometric fluxional calculus, the Cambridge "Analytical Society" together translated Lacroix's *An Elementary Treatise on the Differential and Integral Calculus*, a calculus text that contained, among other material, an algebraic treatment of the calculus drawing on Lagrange's 1797 *Théorie des fonctions analytiques*. Lagrange thought that every function could be expanded into a power series expansion, and its derivative defined purely algebraically. Leibniz's " dx/dy " was not thought of as a quotient, but as "a differential operator" applied to a function. These operators could then be profitably thought of as mathematical objects subject to algebraic manipulation⁸⁶ – even though differential operators are neither numbers nor geometrical magnitudes. This led algebraists to ask just how widely algebraic operations could be applied, and to ask after the reason for their wide applicability. (And these questions would be given a very satisfactory answer if logic itself were a kind of algebra.)

A related conceptual expansion of algebra resulted from the use of negative and imaginary numbers. Peacock's *A Treatise on Algebra* provided a novel justification: he distinguished "arithmetical algebra" from "symbolic algebra" – a strictly formal science of symbols and their combinations, where " $a-b$ " is meaningful even if $a < b$. Facts in arithmetical algebra can be transferred into symbolic algebra by the "principle of the permanence of equivalent forms."⁸⁷ Duncan Gregory defined symbolic algebra as the "science which treats of the combination of operations defined not by their nature, that is, by what they are or what they do, but by the laws of combination to which they are subject."⁸⁸ This is the background to De Morgan's equating the mathematician's notion of form with the operation of a mechanism.

Gregory identifies five different kinds of symbolic algebras.⁸⁹ One algebra is commutative, distributive, and subject to the law $a^m \cdot a^n = a^{m+n}$. George Boole, renaming the third law the "index law," followed Gregory in making these three the fundamental laws of the algebra of differential operators.⁹⁰ Three years later Boole introduced an algebra of logic that obeys these same laws,

use of this notion of form taken from British algebra is – for better or worse – to transform Kant's way of conceiving the formality of logic.

⁸⁶ As did Servois; see F. J. Servois, *Essai sur un nouveau mode d'exposition des principes du calcul différentiel* (Paris: Nismes, 1814).

⁸⁷ George Peacock, *A Treatise on Algebra*, vol. 1, *Arithmetical Algebra*; vol. 2, *On Symbolical Algebra and Its Applications to the Geometry of Position*, 2nd ed. (Cambridge: Cambridge University Press, 1842–5), II 59. (Original edition, 1830.)

⁸⁸ Duncan Gregory, "On the Real Nature of Symbolical Algebra," reprinted in *The Mathematical Works of Duncan Farquharson Gregory*, ed. William Walton (Cambridge: Deighton Bell, 1865), 2. (Original edition, 1838.)

⁸⁹ Gregory, "Symbolical Algebra," 6–7.

⁹⁰ George Boole, "On a General Method in Analysis," *Philosophical Transactions of the Royal Society of London* 134 (1844): 225–82.

with the index law modified to $x^n = x$ for n nonzero.⁹¹ The symbolic algebra defined by these three laws could be interpreted in different ways: as an algebra of the numbers 0 and 1, as an algebra of classes, or as an algebra of propositions.⁹² Understood as classes, ab is the class of things that are in a and in b ; $a + b$ is the class of things that are in a or in b , but not in both; and $a - b$ is the class of things that are in a and not in b ; 0 and 1 are the empty class and the universe. The index law holds (e.g., for $n = 2$) because the class of things that are in both x and x is just x . These three laws are more fundamental than Aristotle's *dictum*;⁹³ in fact, the principle of contradiction is derivable from the index law, since $x - x^2 = x(1 - x) = 0$.⁹⁴

The class of propositions in Boole's algebra – equations with arbitrary numbers of class terms combined by multiplication, addition, and subtraction – is wider than the class amenable to syllogistic, which only handles one subject and one predicate class per proposition.⁹⁵ Just as important, Boole can avoid all of the traditional techniques of conversion, mood, and figure by employing algebraic techniques for the solution of logical equations. Despite the impressive power of Boole's method, it falls well short of modern standards of rigor. The solution of equations generally involves eliminating factors, and so dividing class terms – even though Boole admits that no logical interpretation can be given to division.⁹⁶ But since Boole allows himself to treat logical equations as propositions about the numbers 0 and 1, the interpretation of division is reduced to interpreting the coefficients “1/1,” “1/0,” “0/1,” and “0/0.” Using informal justifications that convinced few, he rejected “1/0” as meaningless and threw out every term with it as a coefficient, and he interpreted “0/0” as referring to some indefinite class “ v .”

Boole, clearly influenced by Peacock, argued that there was no necessity in giving an interpretation to logical division, since the validity of any “symbolic process of reasoning” depends only on the interpretability of the final conclusion.⁹⁷ Jevons thought this an incredible position for a *logician* and discarded division in order to make all results in his system interpretable.⁹⁸ Venn

⁹¹ George Boole, *The Mathematical Analysis of Logic* (Cambridge: Macmillan, Barclay & Macmillan, 1847), 16–18.

⁹² George Boole, *An Investigation of the Laws of Thought* (London: Walton & Maberly, 1854), 37.

⁹³ Boole, *Mathematical Analysis*, 18.

⁹⁴ Boole, *Laws of Thought*, 49.

⁹⁵ Boole, *Laws of Thought*, 238.

⁹⁶ Boole, *Laws of Thought*, 66ff.

⁹⁷ Boole, *Laws of Thought*, 66ff.

⁹⁸ William Stanley Jevons, “Pure Logic or The Logic of Quality Apart from Quantity, with Remarks on Boole's System and on the Relation of Logic to Mathematics,” reprinted in *Pure Logic and Other Minor Works*, eds. Robert Adamson and Harriet A. Jevons (New York: Macmillan, 1890), §174 and §197ff. (Original edition, 1864.)

retained logical division but interpreted it as “logical abstraction” – as had Schröder, whose 1877 book introduced Boolean logic into Germany.⁹⁹ Boole had interpreted disjunction exclusively, allowing him to interpret his equations indifferently as about classes or the algebra of the numbers 0 and 1 (for which “ $x + y$ ” has meaning only if $xy = 0$). Jevons argued that “or” in ordinary language is actually inclusive, and he effected great simplification in his system by introducing the law $A + A = A$.¹⁰⁰

Peirce departs from Boole’s inconvenient practice of only considering equations and introduces a primitive symbol for class inclusion. Peirce also combines in a fruitful way Boole’s algebra with De Morgan’s logic of relations. He conceives of relations (not as copulae, but) as classes of ordered pairs,¹⁰¹ and he introduces Boolean operations on relations. Thus “ $l + s$ ” means “lover or servant.”¹⁰² This research culminated in Peirce’s 1883 “The Logic of Relatives,” which independently of Frege introduced into the Boolean tradition polyadic quantification.¹⁰³ Peirce writes the relative term “lover” as

$$l = \sum_i \sum_j (l)_{ij} (I : J)$$

where “ Σ ” denotes the summation operator, “ $I : J$ ” denotes an ordered pair of individuals, and “ $(l)_{ij}$ ” denotes a coefficient whose value is 1 if I does love J , and 0 otherwise.¹⁰⁴ Then, for instance,

$$\prod_i \sum_j (l)_{ij} > 0$$

means “everybody loves somebody.”¹⁰⁵

GERMAN LOGIC AFTER HEGEL

As post-Kantian idealism waned after Hegel’s death, the most significant German logicians – Trendelenburg, Lotze, Sigwart, and Überweg – came to

⁹⁹ Venn, *Symbolic Logic*, 73ff. Ernst Schröder, *Der Operationskreis des Logikkalküls* (Leipzig: Teubner, 1877), 33.

¹⁰⁰ Jevons, “Pure Logic,” §178, §193; cf. Schröder, *Operationskreis*, 3. Though Jevons’s practice of treating disjunction inclusively has since become standard, Boole’s practice was in keeping with the tradition. Traditional logicians tended to think of disjunction on the model of the relation between different subspecies within a genus, and since two subspecies exclude one another, so too did disjunctive judgments. (See, e.g., Kant, *Critique of Pure Reason*, A73–4/B99.)

¹⁰¹ Charles Sanders Peirce, “Description of a Notation for the Logic of Relatives, Resulting from an Amplification of the Conceptions of Boole’s Calculus of Logic,” reprinted in *Collected Papers of Charles Sanders Peirce*, vol. 3, *Exact Logic*, ed. Charles Hartshorne and Paul Weiss (Cambridge, Mass.: Harvard University Press, 1933), 76. (Original edition, 1870.)

¹⁰² Peirce, “Description,” 38.

¹⁰³ Peirce, “The Logic of Relatives,” reprinted in *Collected Papers*, vol. 3. (Original edition, 1883.)

¹⁰⁴ Peirce, “The Logic of Relatives,” 195.

¹⁰⁵ Peirce, “The Logic of Relatives,” 207.

prefer a middle way between a “subjectively formal” logic and an identification of logic with metaphysics.¹⁰⁶ On this view, logic is not the study of mere thinking or of being itself, but of *knowledge* – a position articulated earlier in Friedrich Schleiermacher’s *Dialektik*.

In lectures given between 1811 and 1833, Schleiermacher calls his “dialectic” the “science of the supreme principles of knowing” and “the art of scientific thinking.”¹⁰⁷ To produce knowledge is an activity, and so the discipline that studies that activity is an art, not a mere canon.¹⁰⁸ This activity is fundamentally social and occurs within a definite historical context. Schleiermacher thus rejects the Fichtean project of founding all knowledge on a first principle; instead, our knowledge always begins “in the middle.”¹⁰⁹ Because individuals acquire knowledge together with other people, dialectic is also – playing up the Socratic meaning – “the art of conversation in pure thinking.”¹¹⁰ Though transcendental and formal philosophy are one,¹¹¹ the principles of being and the principles of knowing are not identical. Rather, there is a kind of parallelism between the two realms. For example, corresponding to the fact that our thinking employs concepts and judgments is the fact that the world is composed of substantial forms standing in systematic causal relations.¹¹²

Trendelenburg, whose enthusiasm for Aristotle’s logic over its modern perversions led him to publish a new edition of Aristotle’s organon for student use,¹¹³ agreed with Schleiermacher that logical principles correspond to, but are not identical with, metaphysical principles. But unlike Schleiermacher,¹¹⁴ Trendelenburg thought that syllogisms are indispensable for laying out the real relations of dependence among things in nature, and he argued that there is a

¹⁰⁶ See Friedrich Überweg, *System of Logic and History of Logical Doctrines*, trans. Thomas M. Lindsay (London: Longmans, Green, 1871), §1, §34. *System der Logik und Geschichte der logischen Lehren*, 3rd ed. (Bonn: Adolph Marcus, 1868). (Original edition, 1857.)

¹⁰⁷ Friedrich Schleiermacher, *Dialectic or The Art of Doing Philosophy*, ed. and trans. T. N. Tice (Atlanta: Scholar’s Press, 1996), 1–5. *Dialektik (1811)*, ed. Andreas Arndt (Hamburg: Felix Meiner, 1986), 3–5, 84. These books contain lecture notes from Schleiermacher’s first lecture course.

¹⁰⁸ Friedrich Schleiermacher, *Dialektik*, ed. L. Jonas as vol. 4 of the second part of *Friedrich Schleiermacher’s sämtliche Werke. Dritte Abtheilung: Zur Philosophie* (Berlin: Reimer, 1839), 364. The Jonas edition of *Dialektik* contains notes and drafts produced between 1811 and 1833. Schleiermacher’s target here is, of course, Kant, who thought that though logic is a set of rules (a “canon”), it is not an “organon” – an instrument for expanding our knowledge.

¹⁰⁹ Schleiermacher, *Dialectic*, 3 (*Dialektik (1811)*, 3). Schleiermacher, *Dialektik*, §291.

¹¹⁰ Schleiermacher, *Dialektik*, §1.

¹¹¹ Schleiermacher, *Dialectic*, 1–2 (*Dialektik (1811)*, 5).

¹¹² Schleiermacher, *Dialektik*, §195.

¹¹³ Adolf Trendelenburg, *Excerpta ex Organo Aristotelis* (Berlin: G. Bethge, 1836).

¹¹⁴ For Schleiermacher, syllogistic is not worth studying, since no new knowledge can arise through syllogisms. See Schleiermacher, *Dialectic*, 36 (*Dialektik (1811)*, 30); Schleiermacher, *Dialektik*, §§327–9.

parallelism between the movement from premise to conclusion and the movements of bodies in nature.¹¹⁵

In 1873 and 1874 Christoph Sigwart and Hermann Lotze produced the two German logic texts that were perhaps most widely read in the last decades of the nineteenth century – a period that saw a real spike in the publication of new logic texts. Sigwart sought to “reconstruct logic from the point of view of methodology.”¹¹⁶ Lotze’s *Logic* begins with an account of how the operations of thought allow a subject to apprehend truths.¹¹⁷ In the current of ideas in the mind, some ideas flow together only because of accidental features of the world; some ideas flow together because the realities that give rise to them are in fact related in a nonaccidental way. It is the task of thought to distinguish these two cases – to “reduce coincidence to coherence” – and it is the task of logic to investigate how the concepts, judgments, and inferences of thought introduce this coherence.¹¹⁸

The debate over the relation between logic and psychology, which had been ongoing since Kant, reached a fever pitch in the *Psychologismus-Streit* of the closing decades of the nineteenth century. The term “psychologism” was coined by the Hegelian J. E. Erdmann to describe the philosophy of Friedrich Beneke.¹¹⁹ Erdmann had earlier argued in his own logical work that Hegel, for whom logic is presuppositionless, had decisively shown that logic in no way depends on psychology – logic is not, as Beneke argued, “applied psychology.”¹²⁰

Contemporary philosophers often associate psychologism with the confusion between laws describing how we do think and laws prescribing how we ought to think.¹²¹ This distinction appears in Kant¹²² and was repeated many times throughout the century.¹²³ The psychologism debate, however, was

¹¹⁵ Trendelenburg, *Untersuchungen*, II 388; I 368.

¹¹⁶ Christoph Sigwart, *Logic*, trans. Helen Dendy, 2 vols. (New York: Macmillan, 1895), I i. *Logik*, 2nd ed., 2 vols. (Freiburg: J. C. B. Mohr, 1889, 1893). (1st ed. of vol. 1, 1873; 2nd ed., 1878.)

¹¹⁷ Hermann Lotze, *Logic*, trans. Bernard Bosanquet, 2 vols. (Oxford: Clarendon Press, 1884), §II. *Logik: Drei Bücher vom Denken, vom Untersuchen und vom Erkennen*, 2nd ed. (Leipzig: S. Hirzel, 1880). (Original edition, 1874.) Whenever possible, I cite by paragraph numbers (§), which are common to the English and German editions.

¹¹⁸ Lotze, *Logic*, §XI.

¹¹⁹ Johann Eduard Erdmann, *Grundriss der Geschichte der Philosophie: Zweiter Band*, 2nd ed. (Berlin: Wilhelm Hertz, 1870), 636.

¹²⁰ Johann Eduard Erdmann, *Outlines of Logic and Metaphysics*, trans. B. C. Burt (New York: Macmillan, 1896), §2. *Grundriss der Logik und Metaphysik*, 4th ed. (Halle: H. W. Schmidt, 1864). (Original edition, 1841.) For a representative passage in Beneke, see *System der Logik als Kunstlehre des Denken: Erster Theil* (Berlin: Ferdinand Dümmler, 1842), 17–18.

¹²¹ See, for example, Gottlob Frege, *Grundgesetze der Arithmetik*, vol. 1 (Jena: H. Pohle, 1893), xv, trans. Michael Beaney in *The Frege Reader* (Oxford: Blackwell, 1997), 202.

¹²² Kant, Ak 9:14.

¹²³ See, for example, Johann Friedrich Herbart, *Psychologie als Wissenschaft* (Königsberg: Unzer, 1825), 173; Boole, *Laws of Thought*, 408–9.

not so much about whether such a distinction could be drawn, but whether this distinction entailed that psychology and logic were independent. Husserl argued that the distinction between descriptive and normative laws was insufficient to protect against psychologism, since the psychologistic logician could maintain, as did Mill, that the science of *correct* reasoning is a branch of psychology, since the laws that describe all thinking surely also apply to the subclass of correct thinking.¹²⁴ Indeed, Mill argues, if logic is to characterize the process of correct thinking, it has to draw on an analysis of how human thinking in fact operates.¹²⁵

Mansel argued that the possibility of logical error in no way affects the character of logic as the science of those “mental laws to which every sound thinker is bound to conform.”¹²⁶ After all, it is only a contingent fact about us that we can make errors and the logical works written by beings for whom logical laws were in fact natural laws would look the same as ours. Sigwart, while granting that logic is the *ethics* and not the *physics* of thinking,¹²⁷ nevertheless argues that taking some kinds of thinking and not others as *normative* can only be justified psychologically – by noting when we experience the “*immediate consciousness* of evident truth.”¹²⁸ Mill also thinks that logical laws are grounded in psychological facts: we infer the principle of contradiction, for instance, from the introspectible fact that “Belief and Disbelief are two different mental states, excluding one another.”¹²⁹

For Lotze, the distinction between truth and “untruth” is absolutely fundamental to logic but of no special concern to psychology. Thus psychology can tell us how we come to believe logical laws, but it cannot ground their truth.¹³⁰ Frege argued in a similar vein that psychology investigates how humans come to hold a logical law to be true but has nothing to say about the law’s being true.¹³¹

¹²⁴ Edmund Husserl, *Logische Untersuchungen. Erster Teil: Prolegomena zur reinen Logik* (Halle: Niemeyer, 1900), §19. Mill, *Examination*, 359.

¹²⁵ Mill, *Logic*, 12–13.

¹²⁶ Mansel, *Prolegomena*, 16.

¹²⁷ Sigwart, *Logic*, I 20 (*Logik*, I 22).

¹²⁸ Sigwart, *Logic*, I §3 (*Logik*, I §3).

¹²⁹ Mill, *Logic*, 277. But Mill appears to equivocate. He elsewhere suggests that the principle is merely verbal after all: Mill, “Grote’s Aristotle,” reprinted in vol. 11 of *Collected Works of John Stuart Mill*, 499–500. (Compare here *Logic*, 178.) Still elsewhere, he seems agnostic whether the principle is grounded in the innate constitution of our mind: *Examination*, 381.

¹³⁰ Lotze, *Logic*, §X; §332.

¹³¹ Gottlob Frege, “Logic,” trans. Peter Lond and Roger White in *Posthumous Writings*, eds. Hans Hermes, Friedrich Kambartel, and Friedrich Kaulbach (Oxford: Basil Blackwell, 1979), 145. “Logik,” in *Nachgelassene Schriften*, eds. Hans Hermes, Friedrich Kambartel, and Friedrich Kaulbach (Hamburg: Felix Meiner, 1969), 157. See also Frege, *The Foundations of Arithmetic*, trans. J. L. Austin (Oxford: Blackwell, 1950), vi. *Die Grundlagen der Arithmetik. Eine logisch-mathematische*

Lotze thus supplemented the antipsychologistic arguments arising from the *normativity* of logic with a separate line of argumentation based on the *objectivity* of the domain of logic. For Lotze, the subjective *act* of thinking is distinct from the objective *content* of thought. This content “presents itself as the same self-identical object to the consciousness of others.”¹³² Logic, but not psychology, concerns itself with the objective relations among the objective thought contents.¹³³ Lotze was surely not the first philosopher to insist on an act/object distinction. Herbart had earlier improved on the ambiguous talk of “concepts” by distinguishing among the act of thinking (the *conceiving*), the concept itself (that which is *conceived*), and the real things that fall under the concept.¹³⁴ But Lotze’s contribution was to use the act/object distinction to secure the objectivity or sharability of thoughts. He interprets and defends Plato’s doctrine of Ideas as affirming that no thinker *creates* or *makes true* the truths that he thinks.¹³⁵

Like Lotze, Frege associated the confusion of logic with psychology with erasing the distinction between the objective and the subjective.¹³⁶ Concepts are not something subjective, such as an idea, because the same concept can be known intersubjectively. Similarly, “we cannot regard thinking as a process which generates thoughts. . . . For do we not say that the same thought is grasped by this person and by that person?”¹³⁷ For Lotze, intersubjective thoughts are somehow still *products* of acts of thinking.¹³⁸ For Frege, though, a thought exists independently of our thinking – it is “independent of our thinking as such.”¹³⁹ Indeed, for Frege any logic that describes the *process* of correct thinking would be psychologistic: logic entirely concerns the most general truths concerning the *contents* (not the acts) of thought.¹⁴⁰

Untersuchung über den Begriff der Zahl (Breslau: W. Koebner, 1884), vi. (Identical paginations in German and English editions.)

¹³² Lotze, *Logic*, §345.

¹³³ Lotze, *Logic*, §332.

¹³⁴ Johann Friedrich Herbart, *Lehrbuch zur Einleitung in die Philosophie*, reprinted in *Johann Friedrich Herbart’s Sämtliche Werke: Erster Band* (Leipzig: Leopold Voss, 1850), §§34–5. (Original edition, 1813.) Independently, Mill severely criticized what he called “conceptualism” – a position common “from Descartes downwards, and especially from the era of Leibniz and Locke” – for confusing the *act* of judging with the thing judged. See Mill, *Logic*, 87–9.

¹³⁵ Lotze, *Logic*, §313ff. On this nonmetaphysical reading of Plato, recognizing the sharability and judgment-independence of truth (§314) does not require hypostasizing the contents of thought or confusing the kind of reality they possess (which Lotze influentially called “validity”) with the existence of things in space and time (§316).

¹³⁶ Frege, *Foundations*, x.

¹³⁷ Frege, *Foundations*, vii, §4. Frege, “Logic,” 137 (*Nachgelassene*, 148–9).

¹³⁸ Lotze, *Logic*, §345.

¹³⁹ Frege, *Frege Reader*, 206 (*Grundgesetze*, I xxiv). Frege, “Logic,” 133 (*Nachgelassene*, 144–5).

¹⁴⁰ Frege, “Logic,” 146 (*Nachgelassene*, 158). Frege, *Frege Reader*, 202 (*Grundgesetze*, I xv).

There are then multiple independent theses that one might call “anti-psychologistic.” One thesis asserts the independence of logic from psychology. Another insists on the distinction between descriptive, psychological laws and normative, logical laws. Another thesis denies that logical laws can be grounded in psychological facts. Yet another thesis denies that logic concerns processes of thinking at all. Still another thesis emphasizes the independence of the truth of thought-contents from acts of holding-true. A stronger thesis maintains the objective existence of thought-contents.

Although Bernard Bolzano’s *Theory of Science* was published in 1837, it was largely unread until the 1890s, when it was rediscovered by some of Brentano’s students.¹⁴¹ Bolzano emphasized more strongly than any thinker before Frege both that the truth of thought-contents is independent of acts of holding-true and that there are objective thought-contents.¹⁴² A “proposition in itself” is “any assertion that something is or is not the case, regardless whether somebody has put it into words, and regardless even whether or not it has been thought.”¹⁴³ It differs both from spoken propositions (which are speech acts) and from mental propositions insofar as the proposition in itself is the *content* of these acts. Propositions and their parts are not real, and they neither exist nor have being; they do not come into being and pass away.¹⁴⁴

THE DOCTRINE OF TERMS

For the remainder of this article, we move from consideration of how the various conceptions of logic evolved throughout the century to an overview of some of the logical topics discussed most widely in the period. According to the tradition, a logic text began with a section on *terms*, moved on to a section on those *judgments* or propositions composed of terms, and ended with a section on *inferences*. Many of the works of the century continued to follow this model, and we will follow suit here.

A fundamental debate among nineteenth-century logicians concerned what the most basic elements of logic are. In the early modern period, Arnauld and

¹⁴¹ Bernard Bolzano, *Theory of Science*, ed. and trans. Rolf George (Berkeley: University of California Press, 1972). The now-standard German edition is Bernard Bolzano, *Wissenschaftslehre*, ed. Jan Berg as vols. 11–14 of ser. 1 of *Bernard-Bolzano-Gesamtausgabe*, eds. Eduard Winter, Jan Berg, Friedrich Kambartel, Jaromír Louzil, and Bob van Rootselaar (Stuttgart: Frommann, 1985–2000). Cited by paragraph numbers (§), common to English and German editions. As an example of Bolzano’s belated recognition, see Husserl, *Logische Untersuchungen*, 225–7.

¹⁴² Bolzano, *Theory of Science*, §48 (*Wissenschaftslehre*, 11.2: §48).

¹⁴³ Bolzano, *Theory of Science*, §19 (*Wissenschaftslehre*, 11.1:104).

¹⁴⁴ Bolzano, *Theory of Science*, §19 (*Wissenschaftslehre*, 11.1:105).

Nicole had transformed the doctrine of terms into the doctrine of *ideas*.¹⁴⁵ Kant, having distinguished intuitions from concepts, restricts the province of logic to *conceptual representations*.¹⁴⁶ Mill, taking seriously that language is the chief instrument of thinking,¹⁴⁷ begins with a discussion of *names*, distinguishing between general and individual, categorematic and syncategorematic, and connotative and nonconnotative (or denotative) names.¹⁴⁸

For Mill, the attribute named by the predicate term in a proposition is affirmed or denied of the object(s) named by the subject term.¹⁴⁹ Mill opposes this view to the common British theory that a proposition expresses that the class picked out by the subject is included in the class picked out by the predicate.¹⁵⁰ Though Mill thinks that “there are as many actual classes (either of real or of imaginary things) as there are general names,”¹⁵¹ he still insists that the use of predicate terms in affirming an attribute of an object is more fundamental and makes possible the formation of classes. The debate over whether a proposition is fundamentally the expression of a relation among classes or a predication of an attribute overlapped with the debate over whether logic should consider terms *extensionally* or *intensionally*.¹⁵² Logicians after Arnauld and Nicole distinguished between the *intension* and the *extension* of a term. The intension (or content) comprises those concepts it contains. The extension of a term is the things contained under it – either its subspecies¹⁵³ or the objects falling under it. Hamilton thought that logic could consider judgments both as the inclusion of the extension of the subject concept in the extension of the predicate concept and as the inclusion of the predicate concept in the

¹⁴⁵ Antoine Arnauld and Pierre Nicole, *Logic or the Art of Thinking*, trans. Jill Vance Buroker (Cambridge: Cambridge University Press, 1996).

¹⁴⁶ Kant, *Logic*, §1. Subsequent German “formal” logicians followed Kant, as did the Kantian formal logicians in midcentury Britain. Thus, both Hamilton (*Logic*, I 13, 75, 131) and Mansel (*Prolegomena*, 22, 32) begin their works by distinguishing concepts from intuitions.

¹⁴⁷ Mill, *Logic*, 19–20. Mill’s procedure draws on Whately, who claimed not to understand what a general idea could be if not a name; see Whately, *Elements*, 12–13, 37.

¹⁴⁸ Many of these distinctions had been made earlier by Whately and – indeed – by Scholastics. See Whately, *Elements*, 81.

¹⁴⁹ Mill, *Logic*, 21, 97.

¹⁵⁰ Mill, *Logic*, 94. For an example of the kind of position Mill is attacking, see Whately, *Elements*, 21, 26.

¹⁵¹ Mill, *Logic*, 122.

¹⁵² Mill’s view actually cuts across the traditional distinction: he thinks that the *intension* of the predicate name is an attribute of the *object* picked out by the subject name. Traditional intensionalists thought that the intension of the predicate term is contained in the intension of the subject term; traditional extensionalists thought that the proposition asserts that the extension of the predicate term includes the extension of the subject term.

¹⁵³ See, e.g., Kant, *Logic*, §7–9. In the critical period, when Kant more thoroughly distinguished relations among concepts from relations among objects, Kant began to talk also of *objects* contained under concepts (e.g., *Critique of Pure Reason*, A137/B176).

intension of the subject concept.¹⁵⁴ But, as Mansel rightly objected, if a judgment is synthetic, the subject class is contained in the predicate class without the predicate's being contained in the content of the subject.¹⁵⁵

Boole self-consciously constructed his symbolic logic entirely extensionally.¹⁵⁶ Though Jevons insisted that the intensional interpretation of terms is logically fundamental,¹⁵⁷ the extensional interpretation won out. Venn summarized the common view when he said that the task of symbolic logic is to find the solution to the following problem: given any number of propositions of various types and containing any number of class terms, find the relations of inclusion or inclusion among each class to the rest.¹⁵⁸

Frege sharply distinguished singular terms from predicates¹⁵⁹ and later even the referents of proper names from the referents of predicates.¹⁶⁰ In the traditional logic, "Socrates is mortal" and "Humans are mortal" were treated in the same way. Thus, for Kant, both "Socrates" and "Human" express concepts; every subjudgmental component of a judgment is a concept.¹⁶¹ Frege therefore also departed from the traditional logic in distinguishing the subordination of one concept to another from the subsumption of an object under a concept. This distinction was not made in the Boolean tradition; for Boole, variables always refer to classes, which are thought of as wholes composed of parts.¹⁶² Thus a class being a union of other classes is not distinguished from a class being composed of its elements.

¹⁵⁴ Hamilton, *Lectures*, I 231–2. This position was also defended by Hamilton's student William Thomson, *An Outline of the Necessary Laws of Thought*, 2nd ed. (London: William Pickering, 1849), 189.

¹⁵⁵ Henry Longueville Mansel, "Recent Extensions of Formal Logic," reprinted in *Letters, Lectures, and Reviews*, ed. Henry W. Chandler (London: John Murray, 1873), 71. (Original edition, 1851.)

¹⁵⁶ "What renders logic possible is the existence in our mind of general notions – our ability . . . from any conceivable collection of objects to separate by a mental act those which belong to the given class and to contemplate them apart from the rest." Boole, *Laws of Thought*, 4.

¹⁵⁷ Jevons, "Pure Logic," §1–5, 11, 17.

¹⁵⁸ Venn, *Symbolic Logic*, xx.

¹⁵⁹ Gottlob Frege, *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* (Halle: L. Nebert, 1879), §9, trans. Michael Beaney in *The Frege Reader*. Citations are by paragraph numbers (§), common to the German and English editions, or by page numbers from the original German edition, which are reproduced in the margins of *The Frege Reader*.

¹⁶⁰ Gottlob Frege, "Funktion und Begriff," in *Kleine Schriften*, ed. Ignacio Angelelli (Darmstadt: Wissenschaftliche Buchgesellschaft, 1967), 125–42, trans. Peter Geach and Brian McGuinness as "Function and Concept" in *Collected Papers on Mathematics, Logic, and Philosophy*, ed. Brian McGuinness (New York: Blackwell, 1984).

¹⁶¹ The Kantian distinction between concepts and intuitions is thus not parallel to Frege's concept/object distinction.

¹⁶² Frege points out that the Booleans failed to make this distinction in his paper "Boole's Logical Calculus and the Concept-script," trans. Peter Lond and Roger White in *Posthumous Writings*, 18. *Nachgelassene Schriften*, 19–20. This essay was written in 1880–1. Unknown to Frege, Bolzano had clearly drawn the distinction forty years earlier – see Bolzano, *Theory of Science*, §66.2, §95 (*Wissenschaftslehre*, 11.2:105–6, 11.3:38–9).

The traditional doctrine of concepts included a discussion of how concepts are formed. On the traditional abstractionist model, concepts are formed by noticing similarities or differences among particulars and abstracting the concept, as the common element.¹⁶³ This model came under severe criticism from multiple directions throughout the century. On the traditional “bottom-up” view, the concept *F* is formed from antecedent representations of particular *F*s, and judgments containing *F* are formed subsequent to the acquisition of the concept. For Frege, the order of priority is reversed: “as opposed to [Boole], I start out with judgments and their contents, and not from concepts. . . . I only allow the formation of concepts to proceed from judgments.”¹⁶⁴ The concepts *4th power* and *4th root of 16* are formed, not by abstraction, but by starting with the judgment “ $2^4 = 16$ ” and replacing in its linguistic expression one or more singular terms by variables. These variables can then be bound by the sign for generality to form the quantified relational expressions that give Frege’s new logic its great expressive power. Since the same judgment can be decomposed in different ways, a thinker can form the judgment “ $2^4 = 16$ ” without noting all of the ways in which the judgment can be decomposed. This in turn explains how the new logic can be epistemically ampliative.¹⁶⁵

Kant had earlier asserted his own kind of “priority thesis,”¹⁶⁶ and the historical connection between Kant and Frege’s priority principles is a complicated one that well illustrates how philosophical and technical questions became intertwined during the century. For Kant, concepts are essentially predicates of possible judgments, because only a judgment is subject to truth or falsity,¹⁶⁷ and thus it is only in virtue of being a predicate in a judgment that concepts relate to objects.¹⁶⁸ Though Kant never explicitly turned this thesis against the theory of concept formation by abstraction, Hegel rightly noted that implicit within Kantian philosophy is a theory opposed to abstractionism.¹⁶⁹ For Kant, just as concepts are related to objects because they can be combined in judgments, intuitions are related to objects because the manifold contained in an intuition is combined according to a certain rule provided by a concept.¹⁷⁰ Thus, the theory of concept formation by abstraction cannot be true in general: the very representation of particulars in intuition already requires the possession of a concept. This Kantian-inspired Hegelian argument was directed

¹⁶³ See, e.g., Kant, *Logic*, §6.

¹⁶⁴ Frege, “Boole’s Calculus,” 16–17 (*Nachgelassene*, 17–19).

¹⁶⁵ Frege, “Boole’s Calculus,” 33–4 (*Nachgelassene*, 36–8). Frege, *Foundations*, §88.

¹⁶⁶ Kant, *Critique of Pure Reason*, A68–9/B93–4.

¹⁶⁷ Kant, *Critique of Pure Reason*, B141–2.

¹⁶⁸ Kant, Ak 4:475.

¹⁶⁹ Hegel, *Science of Logic*, 589 (*Werke* 12:22).

¹⁷⁰ Kant, *Critique of Pure Reason*, A150.

against Hamilton and Mansel in T. H. Green's 1874–5 "Lectures on the Formal Logicians" and against Boolean logic by Robert Adamson.¹⁷¹ (Indeed, these works imported post-Kantian reflections on logic into Britain, setting the stage for the idealist logics that gained prominence later in the century.)

Hegel added a second influential attack on abstractionism. The procedure of comparing, reflecting, and abstracting to form common concepts does not have the resources to discriminate between essential marks and any randomly selected common feature.¹⁷² Trendelenburg took this argument one step further and claimed that it was not just the theory of concept formation, but also the *structure* of concepts in the traditional logic that was preventing it from picking out explanatory concepts. A concept should "contain the ground of the things falling under it."¹⁷³ Thus, the higher concept is to provide the "law" for the lower concept, and – pace Drobisch¹⁷⁴ – a compound concept cannot just be a sum of marks whose structure could be represented using algebraic signs: *human* is not simply *animal* + *rational*.¹⁷⁵

Lotze extended and modified Trendelenburg's idea. For him, the "organic bond"¹⁷⁶ among the component concepts in a compound concept can be modeled "functionally": the content of the whole concept is some nontrivial function of the content of the component concepts.¹⁷⁷ Concepts formed by *interrelating* component universals such as interdependent variables in a function can be explanatory, then, because the dependence of one thing on another is modeled by the functional dependence of component concepts on one another. Lotze thinks that there are in mathematics kinds of inferences more sophisticated than syllogisms¹⁷⁸ and that it is only in these mathematical inferences that the functional interdependence of concepts is exploited.¹⁷⁹

¹⁷¹ Thomas Hill Green, "Lectures on Formal Logicians," in *Works of Thomas Hill Green*, ed. R. L. Nettleship, 3 vols. (London: Longmans, Green, 1886), 165, 171. Adamson, *History*, 117, 122.

¹⁷² Hegel, *Science of Logic*, 588 (Werke 12:21). Again, this argument was imported to Britain much later in the century. See Green, "Lectures," 193; Adamson, *History*, 133–4.

¹⁷³ Trendelenburg, *Untersuchungen*, I 18–19.

¹⁷⁴ Moritz Wilhelm Drobisch, *Neue Darstellung der Logik*, 2nd ed. (Leipzig: Leopold Voss, 1851), ix, §18. Drobisch defended the traditional theory of concept formation against Trendelenburg's attack, occasioning an exchange that continued through the various editions of their works.

¹⁷⁵ Trendelenburg, *Untersuchungen*, I 20. Trendelenburg, "Über Leibnizens Entwurf einer allgemeinen Charakteristik," reprinted in *Historische Beiträge zur Philosophie*, vol. 3 (Berlin: G. Bethge, 1867), 24. (Original edition, 1856.) Trendelenburg thought that his position was Aristotelian. For Aristotle, there is an important metaphysical distinction between a genus and differentia. Yet this distinction is erased when the species-concept is represented, using a commutative operator such as *addition*, as *species* = *genus* + *differentia*.

¹⁷⁶ This is Trendelenburg's Aristotelian language: see Trendelenburg, *Untersuchungen*, I 21.

¹⁷⁷ Lotze, *Logic*, §28.

¹⁷⁸ Lotze, *Logic*, §106ff.

¹⁷⁹ Lotze, *Logic*, §120.

Boolean logic, however, treats concepts as sums and so misses the functionally compound concepts characteristic of mathematics.¹⁸⁰

Frege is thus drawing on a long tradition – springing ultimately from Kant but with significant additions along the way – when he argues that the abstractionist theory of concept formation cannot account for fruitful or explanatory concepts because the “organic” interconnection among component marks in mathematical concepts is not respected when concepts are viewed as sums of marks.¹⁸¹ But Frege *was* the first to think of sentences as analyzable using the function/argument model¹⁸² and the first to appreciate the revolutionary potential of the possibility of multiple decompositionality.

Sigwart gave a still more radical objection to abstractionism. He argued that in order to abstract a concept from a set of representations, we would need some principle for grouping together just this set, and in order to abstract the concept *F* as a common element in the set, we would need already to see them as *F*s. The abstractionist theory is thus circular and presupposes that ability to make judgments containing the concept.¹⁸³

JUDGMENTS AND INFERENCES

The most common objection to Kant’s table of judgments was that he lacked a principle for determining that judgment takes just these forms. Hegel thought of judging as predicating a reflected concept of a being given in sensibility – and so as a relation of thought to being. The truth of a judgment would be the identity of thought and being, of the subject and predicate. Hegel thus tried to explain the completeness of the table of judgments by showing how every form of judgment in which the subject and predicate fail to be completely identical resolves itself into a new form.¹⁸⁴ For Hegel, then, logic acquires systematicity not through reducing the various forms of judgment to one another, but by *deriving* one from another. Other logicians, including those who rejected Hegel’s metaphysics, followed Hegel in trying to derive the various forms from one another.¹⁸⁵

¹⁸⁰ Lotze, *Logic*, I 277ff. (Lotze, *Logik*, 256ff.).

¹⁸¹ Frege, *Foundations*, §88.

¹⁸² Frege, *Begriffsschrift*, vii.

¹⁸³ Sigwart, *Logic*, §40.5. Sigwart thus initiated the widely repeated practice of inverting the traditional organization of logic texts, beginning with a discussion of judgments instead of concepts. Lotze himself thought that Sigwart had gone too far in his objections to abstractionism: Lotze, *Logic*, §8.

¹⁸⁴ Hegel, *Science of Logic*, 625–6, 630 (Werke 54–5, 59).

¹⁸⁵ E.g., Lotze, *Logic*, I 59 (Lotze, *Logik*, 70).

British logicians, on the other hand, tended to underwrite the systematicity of logic by *reducing* the various forms of judgment to one common form. Unlike Kant,¹⁸⁶ Whately reduced disjunctive propositions to hypotheticals in the standard way, and he reduced the hypothetical judgment “If A is B , then X is Y ,” to “The case of A being B is a case of X being Y .”¹⁸⁷ All judgments become categorical, all reasoning becomes syllogizing, and every principle of inference reducible to Aristotle’s *dictum de omni et nullo*. Mansel was more explicit in reading hypothetical judgments temporally: he interprets “If Caius is disengaged, he is writing poetry” as “All times when Caius is disengaged are times when he is writing poetry.”¹⁸⁸ Mill endorses Whately’s procedure, interpreting “If p then q ” as “The proposition q is a legitimate inference from the proposition p .”¹⁸⁹ Mill, of course, thought that syllogistic reasoning is really grounded in induction. But in feeling the need to identify one “universal type of the reasoning process,” Mill was at one with Whately, Hamilton, and Mansel, each of whom tried to reduce induction to syllogisms.¹⁹⁰

Boole notoriously argues that one and the same logical equation can be interpreted either as a statement about classes of things or as a “secondary proposition” – a proposition about other propositions.¹⁹¹ Let “ x ” represent the class of times in which the proposition X is true. Echoing similar proposals by his contemporaries, Boole expresses “If Y then X ” as “ $y = \nu x$ ”: “The time in which Y is true is an indefinite portion of the time in which X is true.”¹⁹² As Frege pointed out, making the calculus of classes and the calculus of propositions two distinct interpretations of the same equations prevents Boole from analyzing the same sentence using quantifiers and sentential operators simultaneously.¹⁹³ Frege’s *Begriffsschrift* gave an axiomatization of truth-functional propositional logic that depends on neither the notion of time nor a calculus of classes. In this, Frege was anticipated by Hugh MacColl.¹⁹⁴ Independently, Peirce axiomatized two-valued truth-functional logic, clearly acknowledging that the material conditional, having no counterfactual meaning, differs from the use of “if” in natural language.¹⁹⁵

¹⁸⁶ Kant, *Critique of Pure Reason*, A73–4/B98–99. Later German logicians tended to follow Kant. See, for example, Fries, *Logik*, §32.

¹⁸⁷ Whately, *Elements*, 71, 74–5.

¹⁸⁸ Mansel, *Prolegomena*, 197.

¹⁸⁹ Mill, *Logic*, 83–4.

¹⁹⁰ Mill, *Logic*, 202. Whately, *Elements*, 153. Hamilton, “Recent Publications,” 162. Mansel, *Prolegomena*, 191.

¹⁹¹ Boole, *Laws of Thought*, 160.

¹⁹² Boole, *Laws of Thought*, 170.

¹⁹³ Frege, “Boole’s Calculus,” 14–15 (*Nachgelassene*, 15–16).

¹⁹⁴ Hugh MacColl, “The Calculus of Equivalent Statements,” *Proceedings of the London Mathematical Society* 9 (1877): 9–20, 177–86.

¹⁹⁵ Charles Sanders Peirce, “On the Algebra of Logic: A Contribution to the Philosophy of Notation,” reprinted in *Collected Papers of Charles Sanders Peirce*, 3:218–19. (Original edition, 1885.)

Among the most discussed and most controversial innovations of the century were Hamilton's and De Morgan's theories of judgments with quantified predicates. In the traditional logic, the quantifier terms "all" and "some" are only applied to the subject term, and so "All *As* are *Bs*" does not distinguish between the case where the *As* are a proper subset of the *Bs* (in Hamilton's language: "All *As* are some *Bs*") and the case where the *As* are coextensive with the *Bs* ("All *As* are all *Bs*"). Now, a distinctive preoccupation of logicians in the modern period was to arrive at systematic methods that would eliminate the need to memorize brute facts about which of the 256 possible cases of the syllogism were valid. A common method was to reduce all syllogisms to the first figure by converting, for example, "All *As* are *Bs*" to "Some *Bs* are *As*."¹⁹⁶ This required students to memorize which judgments were subject to which kinds of conversions. Quantifying the predicate, however, eliminates the distinctions among syllogistic figures and all of the special rules of conversion: all conversion becomes simple – "All *A* is some *B*" is equivalent to "Some *B* is all *A*."¹⁹⁷

Given the simplification allowed by predicate quantification, Hamilton thought he could reduce all of the syllogistic rules to one general canon.¹⁹⁸ De Morgan rightly argued that some of Hamilton's new propositional forms are semantically obscure,¹⁹⁹ and he independently gave his own system and notation for quantified predicates. (Hamilton then initiated a messy dispute over priority and plagiarism with De Morgan.) In De Morgan's notation, there are symbols for the quantity of terms (parentheses), for the negation of the copula (a dot), and – what was new in De Morgan – for the contrary or complement of a class (lowercase letters). "All *As* are (some) *Bs*" is "*A*)*B*." De Morgan gave rules for the interaction of quantification, class contraries, and copula negation.²⁰⁰ The validity of syllogisms is demonstrated very easily, by a simple erasure rule: *Barbara* is "*A*)*B*; *B*)*C*; and so *A*)*C*."²⁰¹

In traditional logic, negation was always attached to the copula "is," and it did not make sense to talk – as De Morgan did – of a negated or contrary term.²⁰² Further departing from tradition, Frege thought of negation as applied to whole sentences and not just to the copula.²⁰³ (Boole, of course, had already

¹⁹⁶ See, e.g., Whately, *Elements*, 61. For an earlier example, see Immanuel Kant, *The False Subtlety of the Four Syllogistic Figures*, in *Theoretical Philosophy: 1755–1770*, eds. and trans. D. Walford and R. Meerbote (Cambridge: Cambridge University Press, 1992).

¹⁹⁷ Hamilton, *Lectures*, II 273.

¹⁹⁸ Hamilton, *Lectures*, II 290–1.

¹⁹⁹ De Morgan, "Logic," 257–8.

²⁰⁰ Augustus De Morgan, *Syllabus of a Proposed System of Logic*, reprinted in *On the Syllogism*, 157ff. (Original edition, 1860.)

²⁰¹ De Morgan, "Syllogism II," 31.

²⁰² As Mansel correctly pointed out: Mansel, "Recent Extensions," 64.

²⁰³ Frege, *Begriffsschrift*, §7.

effectively introduced negation as a sentential operator: he expressed the negation of X as “ \bar{x} .”²⁰⁴ The introduction of contrary terms led De Morgan to restrict possible classes to a background “universe under consideration.” If the universe is specified (say, as living humans), then if A is the class of Britons, a is the class of humans who are not Britons.²⁰⁵

De Morgan’s work on relations led him to distinguish between the relation between two terms and the assertion of that relation, thus separating what was often confused in traditional discussions of the function of the copula.²⁰⁶ But lacking a sign for propositional negation, De Morgan did not explicitly distinguish between negation as a sentential operator and an agent’s denial of a sentence – a mistake not made by Frege.²⁰⁷

De Morgan’s logic of relations was one of many examples of a logical innovation hampered by its adherence to the traditional *subject-copula-predicate* form. Although Bolzano was quite clear about the expressive limitations of the traditional logic in other respects, he nevertheless forced all propositions into the triadic form “*subject has predicate*.”²⁰⁸ In a sense, the decisive move against the *subject-copula-predicate* form was taken by Boole, since an equation can contain an indefinite number of variables, and there is no sense in asking which term is the subject and which is the predicate. Frege went beyond Boole in explicitly recognizing the significance of his break with the subject/predicate analysis of sentences.²⁰⁹

De Morgan required all terms in his system (and their contraries) to be non-empty.²¹⁰ With this requirement, the following nontraditional syllogism turns out valid: “All X s are Y s; all Z s are Y s; therefore, some things are neither X s nor Z s.”²¹¹ Boole, on the other hand, did not assume that the class symbols in his symbolism be nonempty.²¹² The debate over the permissibility of terms with empty extensions dovetailed with longstanding debates over the traditional doctrine that universal affirmative judgments imply particular affirmative

²⁰⁴ Boole, *Laws of Thought*, 168.

²⁰⁵ See De Morgan, *Formal Logic*, 37. Boole adopted De Morgan’s idea, renaming it a “universe of discourse” (*Laws of Thought*, 42).

²⁰⁶ De Morgan, “Syllogism IV,” 215. Compare Mill, *Logic*, 87.

²⁰⁷ Frege, *Begriffsschrift*, §2.

²⁰⁸ Bolzano, *Theory of Science*, §127 (*Wissenschaftslehre*, 12.1:70–1).

²⁰⁹ Frege, *Begriffsschrift*, vii. Frege’s break with the subject-predicate analysis of sentences also made irrelevant the development of systems of quantified predicates among British logicians.

²¹⁰ De Morgan, *Formal Logic*, 127.

²¹¹ De Morgan, “Syllogism II,” 43. This syllogism is valid because – by De Morgan’s requirement that all terms and their contraries be nonempty – there must be some non- Y s, and from the two premises we can infer that the non- Y s cannot be X s or Z s. So some things (namely, the non- Y s) are neither X s nor Z s.

²¹² Boole, *Laws of Thought*, 28.

judgments.²¹³ Herbart denied that the subject term in a judgment “ A is B ” must exist, since (for example) we can judge that the square circle is impossible; Fries argued that “Some griffins are birds” is false, even though “All griffins are birds” is true.²¹⁴ Although Boole himself followed the tradition,²¹⁵ later Booleans tended to follow Herbart and Fries.²¹⁶ Independently Brentano, in keeping with the “reformed logic” made possible largely by his existential theory of judgment, read the universal affirmative as “There is no A that is non- B ” and denied that it implied the particular affirmative.²¹⁷

Whately argued that syllogistic was grounded in one principle only, Aristotle’s *dictum de omni et nullo*: “What is predicated, either affirmatively or negatively, of a term distributed, may be predicated in like manner (affirmatively or negatively) of any thing contained under that term.”²¹⁸ Hamilton thought that the *dictum* was derivable from the more fundamental law: “The part of the part is the part of the whole.”²¹⁹ Mill rejected the *dictum* and identified two principles of the syllogism: “Things which coexist with the same thing, coexist with one another,” and “A thing which coexists with another thing, with which other a third thing does not coexist, is not coexistent with that third thing.” These principles are laws about facts, not ideas, and (he seems to suggest) they are grounded in experience.²²⁰ Mansel argued that the *dictum* could be derived from the more fundamental principles of identity and contradiction, a position taken earlier by Twesten.²²¹ De Morgan argued that the validity of syllogisms depends on the transitivity and commutativity of the copula. He argues against Mansel that these two properties cannot be derived from the principles of contradiction and identity (which gives reflexivity, not commutativity or transitivity).²²²

²¹³ For an example of the traditional view, see Whately, *Elements*, 46–7. Bolzano also defended the tradition: *Theory of Science*, §225 (*Wissenschaftslehre*, 12.3:57–9).

²¹⁴ Herbart, *Lehrbuch*, §53. Fries, *Logik*, 123.

²¹⁵ Boole, *Laws of Thought*, 61. He represented universal affirmatives as “ $x = \nu y$ ” and particular affirmatives as “ $\nu x = \nu y$ ” with ν the symbol for some indefinite selection; given the assumption that νx is always nonzero (*Laws of Thought*, 61) and that $\nu^2 = \nu$, the inference holds (*Laws of Thought*, 229). In general, if a logician allowed for terms with empty extensions, then the inference from universal affirmative to particular affirmative would fail. But Boole illustrates that this holds only in general.

²¹⁶ Charles Sanders Peirce, “On the Algebra of Logic,” reprinted in *Collected Papers of Charles Sanders Peirce*, 3:114. (Original edition, 1880.) Venn, *Symbolic Logic*, 141ff.

²¹⁷ Franz Brentano, *Psychology from an Empirical Standpoint*, trans. A. C. Rancurello, D. B. Terrell, and L. McAlister, ed. Peter Simons, 2nd ed. (London: Routledge, 1995), 230; *Psychologie vom empirischen Standpunkt*, ed. Oskar Kraus, 2nd ed. (Leipzig: Felix Meiner, 1924), II 77. (Original edition, 1874.)

²¹⁸ Whately, *Elements*, 31.

²¹⁹ Hamilton, *Lectures*, I 144–5; compare Kant, *Logic*, §63.

²²⁰ Mill, *Logic*, 178.

²²¹ Mansel, *Prolegomena*, 189. August Twesten, *Die Logik, insbesondere die Analytik* (Schleswig, 1825), §6.

²²² De Morgan, *Formal Logic*, 50ff.; De Morgan, “Syllogism IV,” 214.

By 1860, De Morgan considered syllogistic to be just one material instantiation of the most general form of reasoning: “ A is an L of B ; B is an M of C ; therefore, A is an L of an M of C .” Nevertheless, De Morgan thought that syllogistic needed completing, not discarding: his logic of relations is the “higher atmosphere of the syllogism.”²²³ Hamilton intended his system of quantified predicates to “complete and simplify the old; – to place the keystone in the Aristotelic arch.”²²⁴ Similarly, Venn later argued that Boole’s symbolic logic generalizes and develops the common logic, and he advocates retaining the old syllogistic logic in the classroom.²²⁵ (This attitude contrasts sharply with the contempt for syllogistic shown by German logicians in the generation of Schleiermacher and Hegel.)

Perhaps the deepest and most innovative contribution to the theory of inference was Bolzano’s theory of deducibility, based on the method of idea variation that he introduced in his *Theory of Science*. The propositions $C_p \dots, C_n$ are *deducible* from $P_p \dots, P_m$ with respect to some idea i if every substitution of an idea j for i that makes $P_p \dots, P_m$ true also makes $C_p \dots, C_n$ true.²²⁶ If we restrict our attention to those inferences where the conclusions are deducible with respect to all logical ideas,²²⁷ we isolate a class of “formal” inferences. Bolzano is thus able to pick out all of the logically correct inferences in a fundamentally different way from his contemporaries: he does not try to reduce all possible inferences to one general form, and he does not need to ground the validity of deductions in an overarching principle, like Aristotle’s *dictum* or the principle of identity. Bolzano admits, however, that he has no exhaustive or systematic list of logical ideas – so his idea is not fully worked out.²²⁸

LOGIC, LANGUAGE, AND MATHEMATICS

Debate in Germany over the necessity or possibility of a new logical symbolism centered around Leibniz’s idea of a “universal characteristic,” which was discussed in a widely read paper by Trendelenburg. As Trendelenburg describes the project, Leibniz wanted a language in which, first, the parts of the symbols for a compound concept would be symbols for the parts of the concept itself, and, second, the truth or falsity of any judgment could be determined by calculating.²²⁹ To develop such a language would require first isolating all

²²³ De Morgan, “Syllogism IV,” 241.

²²⁴ Hamilton, *Lectures*, II 251.

²²⁵ Venn, *Symbolic Logic*, xxvii.

²²⁶ Bolzano, *Theory of Science*, §155 (*Wissenschaftslehre*, 12.1:169–86).

²²⁷ Bolzano, *Theory of Science*, §223 (*Wissenschaftslehre*, 12.3:47–9).

²²⁸ Bolzano, *Theory of Science*, §148.2 (*Wissenschaftslehre*, 12.1:141).

²²⁹ Trendelenburg, “Leibnizens Entwurf,” 6, 18.

of the simple concepts or categories.²³⁰ Trendelenburg thought the project was impossible. First, it is not possible to isolate the fundamental concepts of a science before the science is complete, and so the language could not be a tool of scientific discovery.²³¹ Second, Leibniz's characterization of the project presupposes that all concepts can be analyzed as sums of simple concepts, and all reasoning amounts to determining whether one concept is contained in another. Thus, Leibniz's project is subject to all of the objections, posed by Trendelenburg and others, to the abstractionist theory of concept formation and the theory of concepts as sums of marks.²³²

The title of Frege's 1879 book – *Begriffsschrift* or “Concept-script” – is taken from Trendelenburg's essay.²³³ In his 1880 review, Schröder argues that Frege's title does not correspond to the content of the book.²³⁴ A “Begriffsschrift,” or universal characteristic, would require a complete analysis of concepts into basic concepts or “categories” and a proof that the content of every concept can be formed from these categories by a small number of operations. To Schröder, Frege's project is closer to a related Leibnizian project, the development of a *calculus ratiocinator*, a symbolic calculus for carrying out deductive inferences but not for expressing content. In reply, Frege argued that his *begriffsschrift* does differ from Boolean logic in aiming to be both a *calculus ratiocinator* and a universal characteristic.²³⁵ To carry out his logicist project, Frege needs to isolate the axioms of arithmetic, show that these axioms are logical truths and that every concept and object referred to in these axioms is logical, define arithmetical terms, and finally derive the theorems of arithmetic from these axioms and definitions. Since these proofs need to be fully explicit and ordinary language is unacceptably imprecise, it is clear that a logically improved language is needed for expressing the content of arithmetic.²³⁶ Moreover, in strongly rejecting the traditional view that concepts are sums of marks and that all inferring is syllogizing, Frege was answering the objections to Leibniz's project earlier articulated by Trendelenburg.

²³⁰ Trendelenburg, “Leibnizens Entwurf,” 20.

²³¹ Trendelenburg, “Leibnizens Entwurf,” 25.

²³² Trendelenburg, “Leibnizens Entwurf,” 24.

²³³ See Trendelenburg, “Leibnizens Entwurf,” 4.

²³⁴ Ernst Schröder, “Review of Frege's *Begriffsschrift*,” trans. Terrell Ward Bynum in *Conceptual Notation and Related Articles* (Oxford: Oxford University Press, 1972). (Original edition: “Rezension von Gottlob Frege, *Begriffsschrift*,” *Zeitschrift für Mathematik und Physik* 25 [1880]: 81–94.)

²³⁵ Frege, “Boole's Calculus,” 12 (*Nachgelassene*, 13).

²³⁶ Gottlob Frege, “On the Scientific Justification of a Conceptual Notation,” trans. Terrell Ward Bynum in *Conceptual Notation and Related Articles*, 85. (Original edition: “Über die wissenschaftliche Berechtigung einer Begriffsschrift,” *Zeitschrift für Philosophie und philosophische Kritik* 81 [1882]: 50–1.)

Schröder's 1877 *Der Operationskreis des Logikkalküls* opens by arguing that Boole had fulfilled Leibniz's dream of a logical calculus.²³⁷ Later he argued explicitly that his algebra of logic was a necessary and significant step in the development of a universal characteristic or "pasigraphy."²³⁸ Venn, on the other hand, argued that the symbolic logic developed since Boole differs from Leibniz's universal characteristic "as language should and does differ from logic." In symbolic logic, each symbol is a variable standing for any class whatsoever; in a universal characteristic, the symbols refer to definite classes.²³⁹

Frege's thesis that arithmetic is a branch of logic²⁴⁰ was but one contribution in the long debate about the relation between logic and mathematics. An old debate was whether mathematical proofs – specifically, geometrical proofs – could be cast in syllogistic form. Euler had thought so; Schleiermacher did not.²⁴¹ Thomas Reid had argued that an inference involving a judgment with three terms – such as an instance of the transitivity of equality – could not be captured in syllogisms. Hamilton, in his 1846 note in his edition of Reid's works, argues that one can express transitivity of equality syllogistically as "What are equal to the same are equal to each other; A and C are equal to the same (B); therefore, A and C are equal to each other."²⁴² As De Morgan rightly noted, this syllogism does not reduce the transitivity of equality; it presupposes it.²⁴³

Both De Morgan and Boole wanted to make logic symbolic, in a way modeled on mathematics. Mansel accused both Boole and De Morgan of treating logic as an application of mathematics.²⁴⁴ This is a confusion, he contended, because logic is formal and mathematics is material. Boole, unlike De Morgan, took from algebra specific symbols, laws, and methods. Nevertheless, Boole argued

²³⁷ Schröder, *Operationskreis*, iii.

²³⁸ Ernst Schröder, *Vorlesungen über die Algebra der Logik*, vol. 1 (Leipzig: Teubner, 1890), 95. Ernst Schröder, "On Pasigraphy: Its Present State and the Pasigraphic Movement in Italy," *Monist* 9 (1898): 44–62.

²³⁹ Venn, *Symbolic Logic*, 109.

²⁴⁰ Frege, *Grundgesetze*, 1.

²⁴¹ Leonard Euler, *Letters of Euler on Different Subjects in Natural Philosophy: Addressed to a German Princess: with Notes and a Life of Brewster: Containing a Glossary of Scientific Terms with Additional Notes*, by John Grison, trans. Henry Hunter (New York: Harper, 1833), 1:354. (Original edition, 1768–72.) Schleiermacher denied that syllogisms are sufficient for geometrical proofs, since the essential element is the drawing of the additional lines in the diagram: *Dialektik*, 287.

²⁴² Thomas Reid, *Essays on the Active Powers of the Human Mind*, reprinted in *The Works of Thomas Reid, D.D. Now Fully Collected, with Selections from His Unpublished Letters*, with preface, notes, and supplementary dissertations by William Hamilton (Edinburgh: Maclachlan, Stewart, 1846), 702. (Reid's *Essays* first appeared in 1788.) Hamilton's argument appears as an editorial footnote on the same page.

²⁴³ De Morgan, "On the Syllogism II," 67.

²⁴⁴ Mansel, "Recent Extensions," 47. Mansel was directing his argument against Boole's *Laws of Thought* and De Morgan's *Formal Logic*.

that even though logic's "ultimate forms and processes are mathematical," it is only established a posteriori that the algebra of logic can be interpreted indifferently as an algebra of classes or propositions, or again as an algebra of the quantities 0 and 1.²⁴⁵

Jevons accused Boole of, in essence, beginning with self-evident logical notions, transforming them into a symbolism analogous to the algebra of the magnitudes 0 and 1, manipulating the equations as if they were about quantities, and then interpreting them as logical inferences – with no justification save the fact that they seem to work out in the end.²⁴⁶ But this process gets the dependency backward: logic, being purely intensional (or qualitative), is presupposed by the science of number (or quantity), since numbers are composed of qualitatively identical but logically distinct units.²⁴⁷ For Venn, mathematics and symbolic logic are best thought of as two branches of one language of symbols, characterized by a few combinatorial laws. It would be acceptable to think of logic as a branch of mathematics, as long as one understands mathematics – as Boole did – to be "the science of the laws and combinations of symbols."²⁴⁸

Lotze severely criticized Boole for justifying his method on the basis of "rash and misty analogy drawn from the province of mathematics."²⁴⁹ With respect to the relation between the two disciplines, Lotze emphasized that "all calculation is a kind of thought, that the fundamental concepts and principles of mathematics have their systematic place in logic."²⁵⁰ Though some commentators have seen this claim as a forerunner of Frege's logicism,²⁵¹ Lotze means by this something more modest, and yet still very significant. Lotze is advocating that logicians analyze the distinctive kinds of conceptual structures and inferences found in mathematics; such an analysis shows, Lotze thinks, that mathematics outstrips the expressive capacity of syllogistic. Lotze identified three kinds of mathematical inferences irreducible to syllogisms: "inference by substitution," "inference by proportion," and "inference from constitutive equations."²⁵²

²⁴⁵ Boole, *Laws of Thought*, 12; 37.

²⁴⁶ Jevons, "Pure Logic," §202.

²⁴⁷ Jevons, "Pure Logic," §6; §§185–6.

²⁴⁸ Venn, *Symbolic Logic*, xvi–ii.

²⁴⁹ Lotze, *Logic*, I 277–98, especially 278 (*Logik*, 256–69, especially 256).

²⁵⁰ Lotze, "Logic," §18.

²⁵¹ Hans Sluga, *Gottlob Frege* (London: Routledge & Kegan Paul, 1980), 57. Sluga, "Frege: The Early Years," in *Philosophy in History: Essays in the Historiography of Philosophy*, eds. Richard Rorty, Jerome B. Schneewind, and Quentin Skinner (Cambridge: Cambridge University Press, 1984), 343–4. Gottfried Gabriel has argued for a similar conclusion; see his "Objektivität: Logik und Erkenntnistheorie bei Lotze und Frege," in Hermann Lotze, *Logik: Drittes Buch. Vom Erkennen*, ed. Gottfried Gabriel (Hamburg: Felix Meiner, 1989), ix–xxxvi.

²⁵² Lotze, *Logic*, §§105–19.

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