From Triangulations to 4-Manifolds
In Honor of Takao Matumoto's 60^{th} Birthday

Ron Stern

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February 18, 2006

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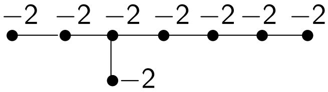
$W =$ 

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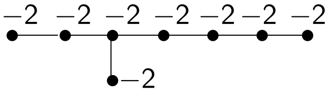
  $W =$ 

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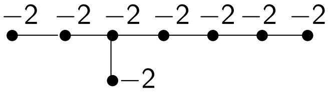
$W =$ 

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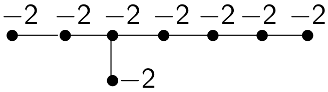
$W =$ 

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- Note that the Poincaré conjecture implies that a 4-manifold is triangulable iff smoothable.

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- $\ker(\mu)$ is infinitely generated (Furuta, Fintushel-Stern 1990 using Donaldson, 1982). Each generator has infinite order.

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- However, these results brought much interest from topologists in low-dimensional topology.

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- Are these infinities related?
- E.g. Fix M^4, then $[\Sigma] \in \ker(\mu)$ yields a smooth structure S_Σ on M^4 and $S_\Sigma = S_{\Sigma'}$ iff $[\Sigma] = [\Sigma'] \in \ker(\mu)$

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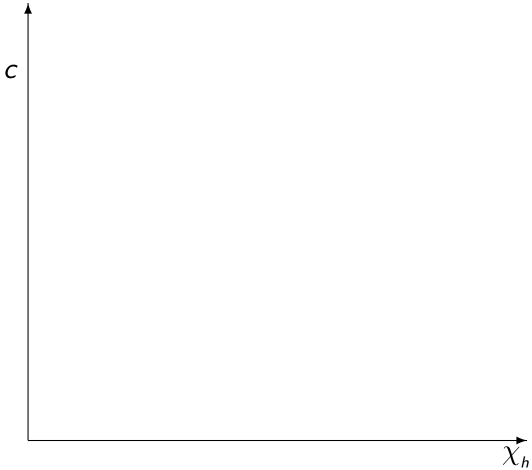

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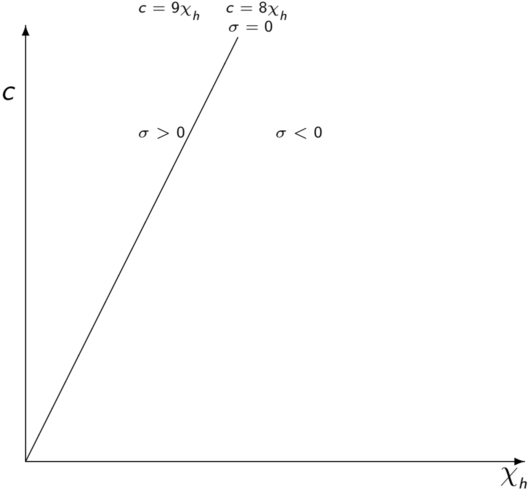

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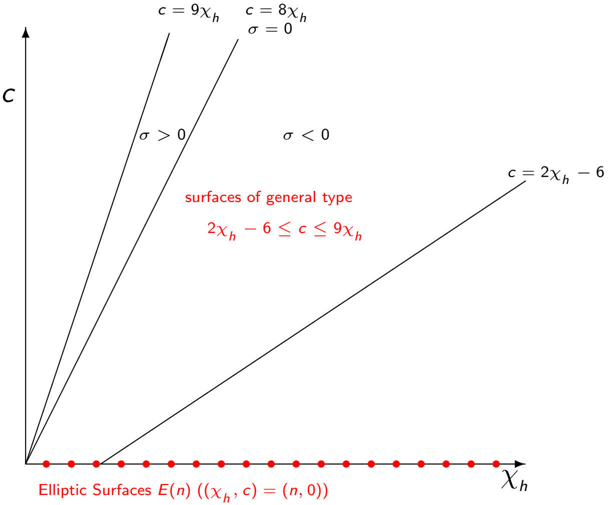

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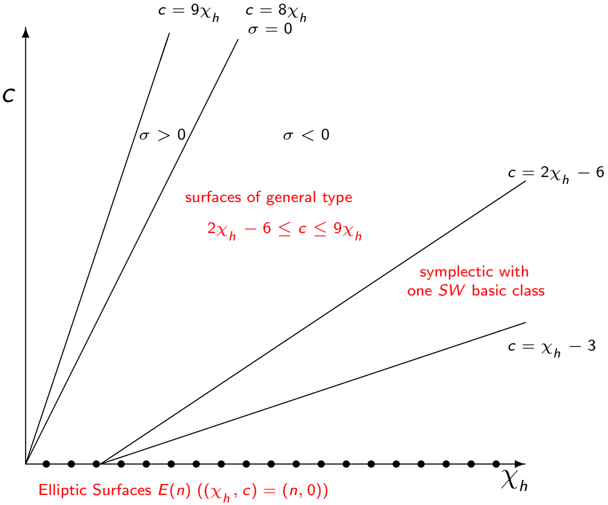

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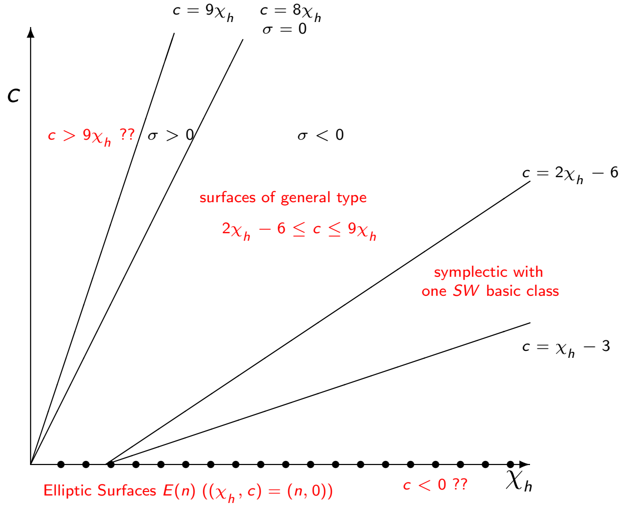

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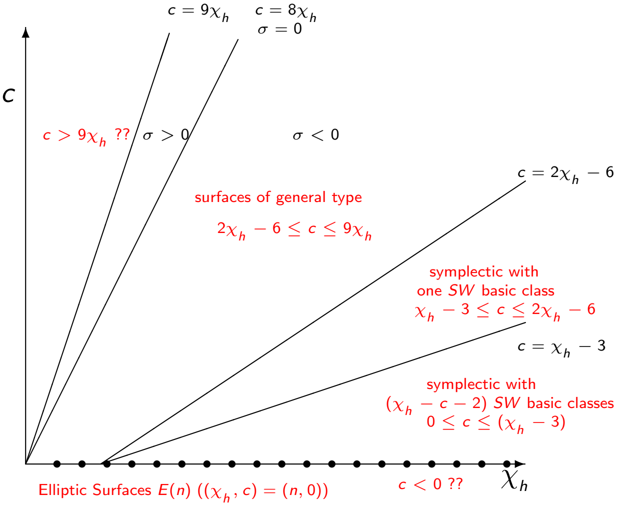

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- All known M^4 with $\chi_h = 1$, $c > 0$, and $\mathcal{SW} \neq 0$ cannot have such tori.

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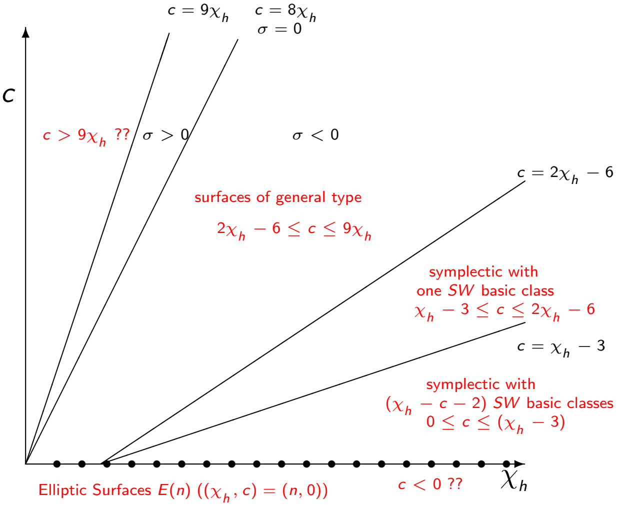

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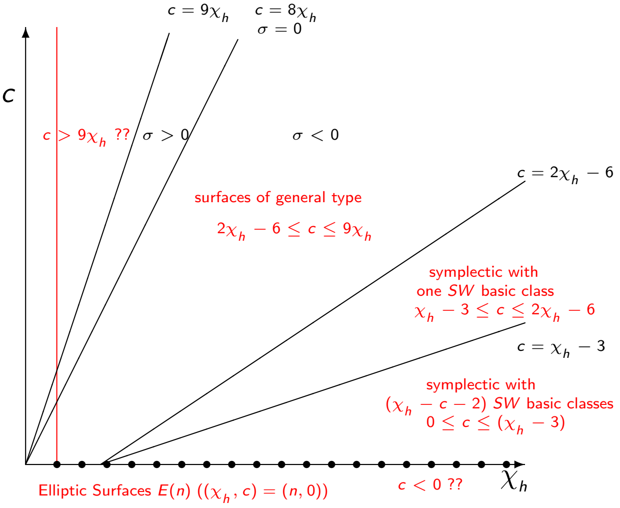

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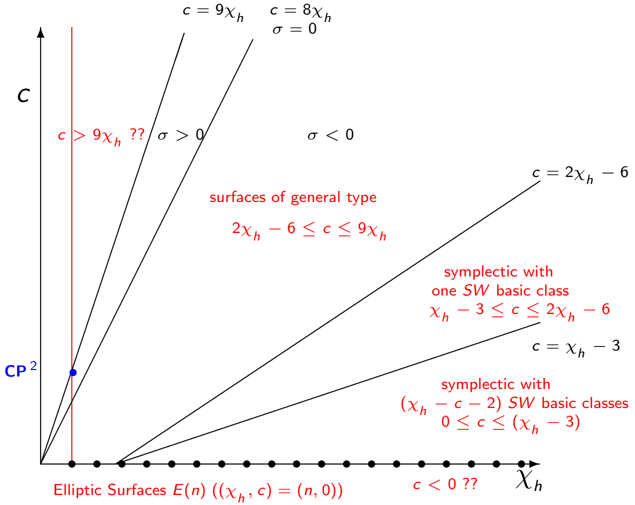

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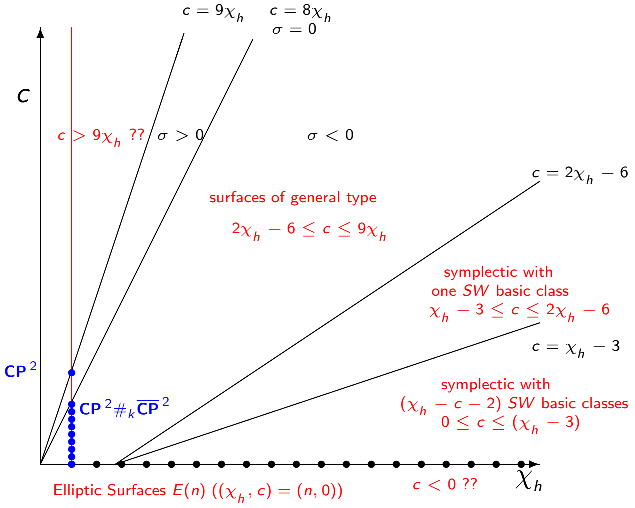

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- What about more than one smooth structure?

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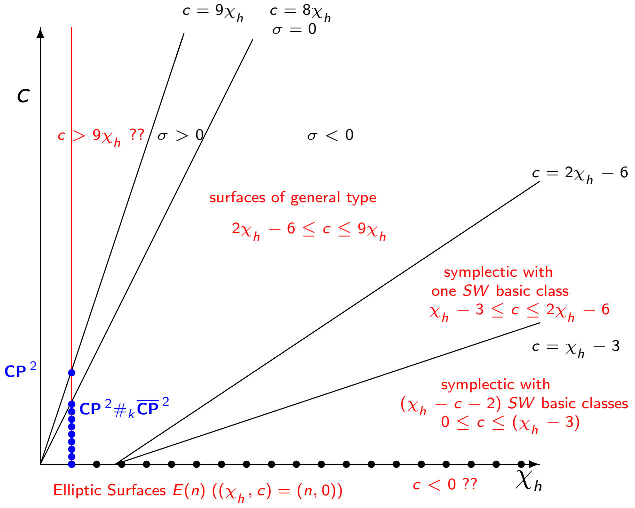

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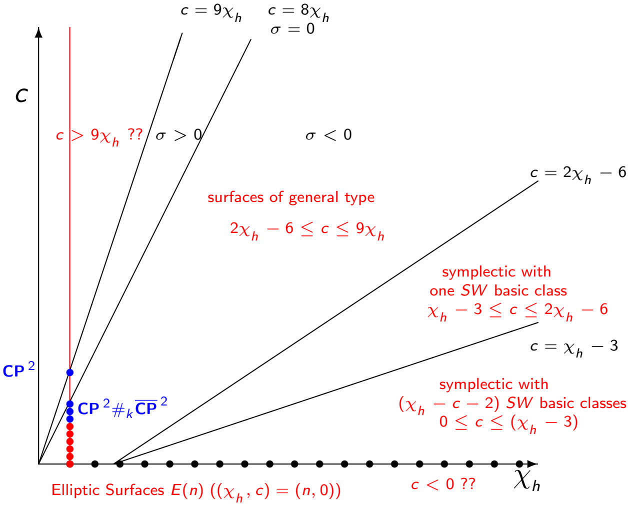

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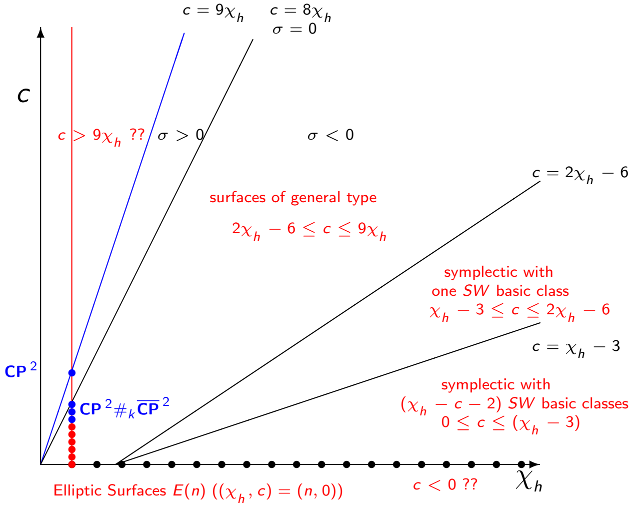

Happy Birthday Takao!